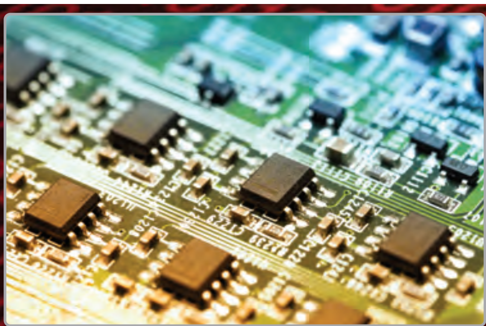
4 Solving Quadratic Equations

4.1 Properties of Radicals
4.2 Solving Quadratic Equations by Graphing
4.3 Solving Quadratic Equations Using Square Roots
4.4 Solving Quadratic Equations by Completing the Square
4.5 Solving Quadratic Equations Using the Quadratic Formula
4.6 Complex Numbers
4.7 Solving Quadratic Equations with Complex Solutions
4.8 Solving Nonlinear Systems of Equations
4.9 Quadratic Inequalities

Robot-Building Competition ***(p. 265)***

Feeding Gannet ***(p. 250)***



Electrical Circuits ***(p. 240)***

Half-pipe ***(p. 223)***

Parthenon ***(p. 195)***

Maintaining Mathematical Proficiency

Factoring Perfect Square Trinomials

Example 1 **Factor $x^2 + 14x + 49$.**

$$x^2 + 14x + 49 = x^2 + 2(x)(7) + 7^2 \qquad \text{Write as } a^2 + 2ab + b^2.$$

$$= (x + 7)^2 \qquad \text{Perfect square trinomial pattern}$$

Factor the trinomial.

1. $x^2 + 10x + 25$

2. $x^2 - 20x + 100$

3. $x^2 + 12x + 36$

4. $x^2 - 18x + 81$

5. $x^2 + 16x + 64$

6. $x^2 - 30x + 225$

Solving Systems of Linear Equations by Graphing

Example 2 **Solve the system of linear equations by graphing.**

$y = 2x + 1$ Equation 1

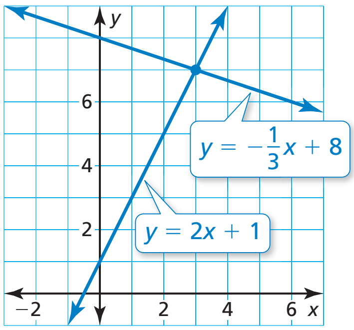
$y = -\frac{1}{3}x + 8$ Equation 2

Step 1 Graph each equation.

Step 2 Estimate the point of intersection.
The graphs appear to intersect at (3, 7).

Step 3 Check your point from Step 2.

Equation 1	Equation 2
$y = 2x + 1$	$y = -\frac{1}{3}x + 8$
$7 \stackrel{?}{=} 2(3) + 1$	$7 \stackrel{?}{=} -\frac{1}{3}(3) + 8$
$7 = 7$ ✓	$7 = 7$ ✓



▶ The solution is (3, 7).

Solve the system of linear equations by graphing.

7. $y = -5x + 3$
$y = 2x - 4$

8. $y = \frac{3}{2}x - 2$
$y = -\frac{1}{4}x + 5$

9. $y = \frac{1}{2}x + 4$
$y = -3x - 3$

10. **ABSTRACT REASONING** What value of c makes $x^2 + bx + c$ a perfect square trinomial?

Mathematical Practices

Mathematically proficient students monitor their work and change course as needed.

Problem-Solving Strategies

Core Concept

Guess, Check, and Revise

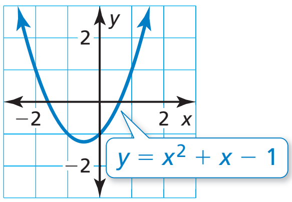
When solving a problem in mathematics, it is often helpful to estimate a solution and then observe how close that solution is to being correct. For instance, you can use the guess, check, and revise strategy to find a decimal approximation of the square root of 2.

	Guess	Check	How to revise
1.	1.4	$1.4^2 = 1.96$	Increase guess.
2.	1.41	$1.41^2 = 1.9881$	Increase guess.
3.	1.415	$1.415^2 = 2.002225$	Decrease guess.

By continuing this process, you can determine that the square root of 2 is approximately 1.4142.

EXAMPLE 1 Approximating a Solution of an Equation

The graph of $y = x^2 + x - 1$ is shown. Approximate the positive solution of the equation $x^2 + x - 1 = 0$ to the nearest thousandth.



SOLUTION

Using the graph, you can make an initial estimate of the positive solution to be $x = 0.65$.

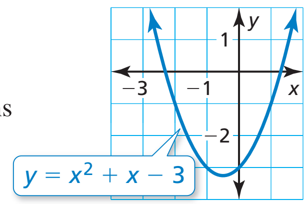
	Guess	Check	How to revise
1.	0.65	$0.65^2 + 0.65 - 1 = 0.0725$	Decrease guess.
2.	0.62	$0.62^2 + 0.62 - 1 = 0.0044$	Decrease guess.
3.	0.618	$0.618^2 + 0.618 - 1 = -0.000076$	Increase guess.
4.	0.6181	$0.6181^2 + 0.6181 - 1 \approx 0.00015$	The solution is between 0.618 and 0.6181.

▶ So, to the nearest thousandth, the positive solution of the equation is $x = 0.618$.

Monitoring Progress

1. Use the graph in Example 1 to approximate the negative solution of the equation $x^2 + x - 1 = 0$ to the nearest thousandth.

2. The graph of $y = x^2 + x - 3$ is shown. Approximate both solutions of the equation $x^2 + x - 3 = 0$ to the nearest thousandth.



4.1 Properties of Radicals

Essential Question

How can you multiply and divide square roots?

EXPLORATION 1 Operations with Square Roots

Work with a partner. For each operation with square roots, compare the results obtained using the two indicated orders of operations. What can you conclude?

a. Square Roots and Addition

Is $\sqrt{36} + \sqrt{64}$ equal to $\sqrt{36 + 64}$?

In general, is $\sqrt{a} + \sqrt{b}$ equal to $\sqrt{a + b}$? Explain your reasoning.

b. Square Roots and Multiplication

Is $\sqrt{4} \cdot \sqrt{9}$ equal to $\sqrt{4 \cdot 9}$?

In general, is $\sqrt{a} \cdot \sqrt{b}$ equal to $\sqrt{a \cdot b}$? Explain your reasoning.

c. Square Roots and Subtraction

Is $\sqrt{64} - \sqrt{36}$ equal to $\sqrt{64 - 36}$?

In general, is $\sqrt{a} - \sqrt{b}$ equal to $\sqrt{a - b}$? Explain your reasoning.

d. Square Roots and Division

Is $\dfrac{\sqrt{100}}{\sqrt{4}}$ equal to $\sqrt{\dfrac{100}{4}}$?

In general, is $\dfrac{\sqrt{a}}{\sqrt{b}}$ equal to $\sqrt{\dfrac{a}{b}}$? Explain your reasoning.

REASONING ABSTRACTLY

To be proficient in math, you need to recognize and use counterexamples.

EXPLORATION 2 Writing Counterexamples

Work with a partner. A **counterexample** is an example that proves that a general statement is *not* true. For each general statement in Exploration 1 that is not true, write a counterexample different from the example given.

Communicate Your Answer

3. How can you multiply and divide square roots?

4. Give an example of multiplying square roots and an example of dividing square roots that are different from the examples in Exploration 1.

5. Write an algebraic rule for each operation.

a. the product of square roots

b. the quotient of square roots

4.1 Lesson

Core Vocabulary

counterexample, *p. 191*
radical expression, *p. 192*
simplest form of a radical, *p. 192*
rationalizing the denominator, *p. 194*
conjugates, *p. 194*
like radicals, *p. 196*

Previous
radicand
perfect cube

What You Will Learn

- Use properties of radicals to simplify expressions.
- Simplify expressions by rationalizing the denominator.
- Perform operations with radicals.

Using Properties of Radicals

A **radical expression** is an expression that contains a radical. An expression involving a radical with index n is in **simplest form** when these three conditions are met.

- No radicands have perfect nth powers as factors other than 1.
- No radicands contain fractions.
- No radicals appear in the denominator of a fraction.

You can use the property below to simplify radical expressions involving square roots.

Core Concept

Product Property of Square Roots

Words The square root of a product equals the product of the square roots of the factors.

Numbers $\sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$

Algebra $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, where $a, b \geq 0$

STUDY TIP

There can be more than one way to factor a radicand. An efficient method is to find the greatest perfect square factor.

EXAMPLE 1 Using the Product Property of Square Roots

a. $\sqrt{108} = \sqrt{36 \cdot 3}$ Factor using the greatest perfect square factor.

$= \sqrt{36} \cdot \sqrt{3}$ Product Property of Square Roots

$= 6\sqrt{3}$ Simplify.

b. $\sqrt{9x^3} = \sqrt{9 \cdot x^2 \cdot x}$ Factor using the greatest perfect square factor.

$= \sqrt{9} \cdot \sqrt{x^2} \cdot \sqrt{x}$ Product Property of Square Roots

$= 3x\sqrt{x}$ Simplify.

STUDY TIP

In this course, whenever a variable appears in the radicand, assume that it has only *nonnegative* values.

Monitoring Progress

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Simplify the expression.

1. $\sqrt{24}$ **2.** $-\sqrt{80}$ **3.** $\sqrt{49x^3}$ **4.** $\sqrt{75n^5}$

Core Concept

Quotient Property of Square Roots

Words The square root of a quotient equals the quotient of the square roots of the numerator and denominator.

Numbers $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$ **Algebra** $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, where $a \geq 0$ and $b > 0$

EXAMPLE 2 Using the Quotient Property of Square Roots

a. $\sqrt{\dfrac{15}{64}} = \dfrac{\sqrt{15}}{\sqrt{64}}$ Quotient Property of Square Roots

$= \dfrac{\sqrt{15}}{8}$ Simplify.

b. $\sqrt{\dfrac{81}{x^2}} = \dfrac{\sqrt{81}}{\sqrt{x^2}}$ Quotient Property of Square Roots

$= \dfrac{9}{x}$ Simplify.

You can extend the Product and Quotient Properties of Square Roots to other radicals, such as cube roots. When using these *properties of cube roots*, the radicands may contain negative numbers.

EXAMPLE 3 Using Properties of Cube Roots

STUDY TIP

To write a cube root in simplest form, find factors of the radicand that are perfect cubes.

a. $\sqrt[3]{-128} = \sqrt[3]{-64 \cdot 2}$ Factor using the greatest perfect cube factor.

$= \sqrt[3]{-64} \cdot \sqrt[3]{2}$ Product Property of Cube Roots

$= -4\sqrt[3]{2}$ Simplify.

b. $\sqrt[3]{125x^7} = \sqrt[3]{125 \cdot x^6 \cdot x}$ Factor using the greatest perfect cube factors.

$= \sqrt[3]{125} \cdot \sqrt[3]{x^6} \cdot \sqrt[3]{x}$ Product Property of Cube Roots

$= 5x^2\sqrt[3]{x}$ Simplify.

c. $\sqrt[3]{\dfrac{y}{216}} = \dfrac{\sqrt[3]{y}}{\sqrt[3]{216}}$ Quotient Property of Cube Roots

$= \dfrac{\sqrt[3]{y}}{6}$ Simplify.

d. $\sqrt[3]{\dfrac{8x^4}{27y^3}} = \dfrac{\sqrt[3]{8x^4}}{\sqrt[3]{27y^3}}$ Quotient Property of Cube Roots

$= \dfrac{\sqrt[3]{8 \cdot x^3 \cdot x}}{\sqrt[3]{27 \cdot y^3}}$ Factor using the greatest perfect cube factors.

$= \dfrac{\sqrt[3]{8} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{x}}{\sqrt[3]{27} \cdot \sqrt[3]{y^3}}$ Product Property of Cube Roots

$= \dfrac{2x\sqrt[3]{x}}{3y}$ Simplify.

Monitoring Progress

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Simplify the expression.

5. $\sqrt{\dfrac{23}{9}}$ **6.** $-\sqrt{\dfrac{17}{100}}$ **7.** $\sqrt{\dfrac{36}{z^2}}$ **8.** $\sqrt{\dfrac{4x^2}{64}}$

9. $\sqrt[3]{54}$ **10.** $\sqrt[3]{16x^4}$ **11.** $\sqrt[3]{\dfrac{a}{-27}}$ **12.** $\sqrt[3]{\dfrac{25c^7d^3}{64}}$

Rationalizing the Denominator

When a radical is in the denominator of a fraction, you can multiply the fraction by an appropriate form of 1 to eliminate the radical from the denominator. This process is called **rationalizing the denominator**.

EXAMPLE 4 Rationalizing the Denominator

a. $\dfrac{\sqrt{5}}{\sqrt{3n}} = \dfrac{\sqrt{5}}{\sqrt{3n}} \cdot \dfrac{\sqrt{3n}}{\sqrt{3n}}$ Multiply by $\dfrac{\sqrt{3n}}{\sqrt{3n}}$.

$= \dfrac{\sqrt{15n}}{\sqrt{9n^2}}$ Product Property of Square Roots

$= \dfrac{\sqrt{15n}}{\sqrt{9} \cdot \sqrt{n^2}}$ Product Property of Square Roots

$= \dfrac{\sqrt{15n}}{3n}$ Simplify.

b. $\dfrac{2}{\sqrt[3]{9}} = \dfrac{2}{\sqrt[3]{9}} \cdot \dfrac{\sqrt[3]{3}}{\sqrt[3]{3}}$ Multiply by $\dfrac{\sqrt[3]{3}}{\sqrt[3]{3}}$.

$= \dfrac{2\sqrt[3]{3}}{\sqrt[3]{27}}$ Product Property of Cube Roots

$= \dfrac{2\sqrt[3]{3}}{3}$ Simplify.

STUDY TIP

Rationalizing the denominator works because you multiply the numerator and denominator by the same nonzero number a, which is the same as multiplying by $\frac{a}{a}$, or 1.

The binomials $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where a, b, c, and d are rational numbers, are called **conjugates**. You can use conjugates to simplify radical expressions that contain a sum or difference involving square roots in the denominator.

EXAMPLE 5 Rationalizing the Denominator Using Conjugates

Simplify $\dfrac{7}{2 - \sqrt{3}}$.

SOLUTION

$\dfrac{7}{2 - \sqrt{3}} = \dfrac{7}{2 - \sqrt{3}} \cdot \dfrac{2 + \sqrt{3}}{2 + \sqrt{3}}$ The conjugate of $2 - \sqrt{3}$ is $2 + \sqrt{3}$.

$= \dfrac{7(2 + \sqrt{3})}{2^2 - (\sqrt{3})^2}$ Sum and difference pattern

$= \dfrac{14 + 7\sqrt{3}}{1}$ Simplify.

$= 14 + 7\sqrt{3}$ Simplify.

LOOKING FOR STRUCTURE

Notice that the product of two conjugates $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$ does not contain a radical and is a *rational* number.

$$\left(a\sqrt{b} + c\sqrt{d}\right)\left(a\sqrt{b} - c\sqrt{d}\right) = \left(a\sqrt{b}\right)^2 - \left(c\sqrt{d}\right)^2 = a^2b - c^2d$$

Monitoring Progress

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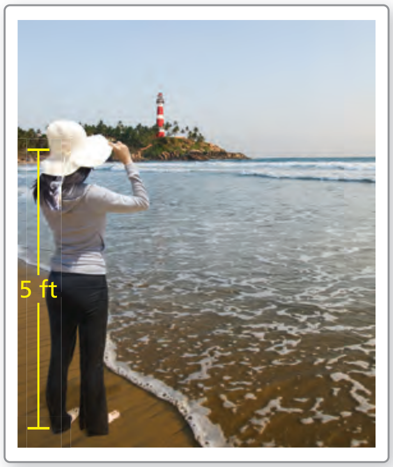
Simplify the expression.

13. $\dfrac{1}{\sqrt{5}}$ **14.** $\dfrac{\sqrt{10}}{\sqrt{3}}$ **15.** $\dfrac{7}{\sqrt{2x}}$ **16.** $\sqrt{\dfrac{2y^2}{3}}$

17. $\dfrac{5}{\sqrt[3]{32}}$ **18.** $\dfrac{8}{1 + \sqrt{3}}$ **19.** $\dfrac{\sqrt{13}}{\sqrt{5} - 2}$ **20.** $\dfrac{12}{\sqrt{2} + \sqrt{7}}$

EXAMPLE 6 Solving a Real-Life Problem



The distance d (in miles) that you can see to the horizon with your eye level h feet above the water is given by $d = \sqrt{\frac{3h}{2}}$. How far can you see when your eye level is 5 feet above the water?

SOLUTION

$$d = \sqrt{\frac{3(5)}{2}} \qquad \text{Substitute 5 for } h.$$

$$= \frac{\sqrt{15}}{\sqrt{2}} \qquad \text{Quotient Property of Square Roots}$$

$$= \frac{\sqrt{15}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \qquad \text{Multiply by } \frac{\sqrt{2}}{\sqrt{2}}.$$

$$= \frac{\sqrt{30}}{2} \qquad \text{Simplify.}$$

▶ You can see $\frac{\sqrt{30}}{2}$, or about 2.74 miles.

EXAMPLE 7 Modeling with Mathematics

The ratio of the length to the width of a *golden rectangle* is $(1 + \sqrt{5}) : 2$. The dimensions of the face of the Parthenon in Greece form a golden rectangle. What is the height h of the Parthenon?

SOLUTION

1. **Understand the Problem** Think of the length and height of the Parthenon as the length and width of a golden rectangle. The length of the rectangular face is 31 meters. You know the ratio of the length to the height. Find the height h.

2. **Make a Plan** Use the ratio $(1 + \sqrt{5}) : 2$ to write a proportion and solve for h.

3. **Solve the Problem**

$$\frac{1 + \sqrt{5}}{2} = \frac{31}{h} \qquad \text{Write a proportion.}$$

$$h(1 + \sqrt{5}) = 62 \qquad \text{Cross Products Property}$$

$$h = \frac{62}{1 + \sqrt{5}} \qquad \text{Divide each side by } 1 + \sqrt{5}.$$

$$h = \frac{62}{1 + \sqrt{5}} \cdot \frac{1 - \sqrt{5}}{1 - \sqrt{5}} \qquad \text{Multiply the numerator and denominator by the conjugate.}$$

$$h = \frac{62 - 62\sqrt{5}}{-4} \qquad \text{Simplify.}$$

$$h \approx 19.16 \qquad \text{Use a calculator.}$$

▶ The height is about 19 meters.

4. **Look Back** $\frac{1 + \sqrt{5}}{2} \approx 1.62$ and $\frac{31}{19.16} \approx 1.62$. So, your answer is reasonable.

Monitoring Progress

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21. **WHAT IF?** In Example 6, how far can you see when your eye level is 35 feet above the water?

22. The dimensions of a dance floor form a golden rectangle. The shorter side of the dance floor is 50 feet. What is the length of the longer side of the dance floor?

Performing Operations with Radicals

Radicals with the same index and radicand are called **like radicals**. You can add and subtract like radicals the same way you combine like terms by using the Distributive Property.

STUDY TIP

Do not assume that radicals with different radicands cannot be added or subtracted. Always check to see whether you can simplify the radicals. In some cases, the radicals will become like radicals.

EXAMPLE 8 Adding and Subtracting Radicals

a. $5\sqrt{7} + \sqrt{11} - 8\sqrt{7} = 5\sqrt{7} - 8\sqrt{7} + \sqrt{11}$ — Commutative Property of Addition

$= (5 - 8)\sqrt{7} + \sqrt{11}$ — Distributive Property

$= -3\sqrt{7} + \sqrt{11}$ — Subtract.

b. $10\sqrt{5} + \sqrt{20} = 10\sqrt{5} + \sqrt{4 \cdot 5}$ — Factor using the greatest perfect square factor.

$= 10\sqrt{5} + \sqrt{4} \cdot \sqrt{5}$ — Product Property of Square Roots

$= 10\sqrt{5} + 2\sqrt{5}$ — Simplify.

$= (10 + 2)\sqrt{5}$ — Distributive Property

$= 12\sqrt{5}$ — Add.

c. $6\sqrt[3]{x} + 2\sqrt[3]{x} = (6 + 2)\sqrt[3]{x}$ — Distributive Property

$= 8\sqrt[3]{x}$ — Add.

EXAMPLE 9 Multiplying Radicals

Simplify $\sqrt{5}(\sqrt{3} - \sqrt{75})$.

SOLUTION

Method 1 $\sqrt{5}(\sqrt{3} - \sqrt{75}) = \sqrt{5} \cdot \sqrt{3} - \sqrt{5} \cdot \sqrt{75}$ — Distributive Property

$= \sqrt{15} - \sqrt{375}$ — Product Property of Square Roots

$= \sqrt{15} - 5\sqrt{15}$ — Simplify.

$= (1 - 5)\sqrt{15}$ — Distributive Property

$= -4\sqrt{15}$ — Subtract.

Method 2 $\sqrt{5}(\sqrt{3} - \sqrt{75}) = \sqrt{5}(\sqrt{3} - 5\sqrt{3})$ — Simplify $\sqrt{75}$.

$= \sqrt{5}[(1 - 5)\sqrt{3}]$ — Distributive Property

$= \sqrt{5}(-4\sqrt{3})$ — Subtract.

$= -4\sqrt{15}$ — Product Property of Square Roots

Monitoring Progress

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Simplify the expression.

23. $3\sqrt{2} - \sqrt{6} + 10\sqrt{2}$

24. $4\sqrt{7} - 6\sqrt{63}$

25. $4\sqrt[3]{5x} - 11\sqrt[3]{5x}$

26. $\sqrt{3}(8\sqrt{2} + 7\sqrt{32})$

27. $(2\sqrt{5} - 4)^2$

28. $\sqrt[3]{-4}(\sqrt[3]{2} - \sqrt[3]{16})$

4.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The process of eliminating a radical from the denominator of a radical expression is called ________________.

2. **VOCABULARY** What is the conjugate of the binomial $\sqrt{6} + 4$?

3. **WRITING** Are the expressions $\frac{1}{3}\sqrt{2x}$ and $\sqrt{\frac{2x}{9}}$ equivalent? Explain your reasoning.

4. **WHICH ONE DOESN'T BELONG?** Which expression does *not* belong with the other three? Explain your reasoning.

$-\frac{1}{3}\sqrt{6}$	$6\sqrt{3}$	$\frac{1}{6}\sqrt{3}$	$-3\sqrt{3}$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, determine whether the expression is in simplest form. If the expression is not in simplest form, explain why.

5. $\sqrt{19}$
6. $\sqrt{\frac{1}{7}}$
7. $\sqrt{48}$
8. $\sqrt{34}$
9. $\frac{5}{\sqrt{2}}$
10. $\frac{3\sqrt{10}}{4}$
11. $\frac{1}{2 + \sqrt[3]{2}}$
12. $6 - \sqrt[3]{54}$

In Exercises 13–20, simplify the expression. *(See Example 1.)*

13. $\sqrt{20}$
14. $\sqrt{32}$
15. $\sqrt{128}$
16. $-\sqrt{72}$
17. $\sqrt{125b}$
18. $\sqrt{4x^2}$
19. $-\sqrt{81m^3}$
20. $\sqrt{48n^5}$

In Exercises 21–28, simplify the expression. *(See Example 2.)*

21. $\sqrt{\frac{4}{49}}$
22. $-\sqrt{\frac{7}{81}}$
23. $-\sqrt{\frac{23}{64}}$
24. $\sqrt{\frac{65}{121}}$
25. $\sqrt{\frac{a^3}{49}}$
26. $\sqrt{\frac{144}{k^2}}$
27. $\sqrt{\frac{100}{4x^2}}$
28. $\sqrt{\frac{25v^2}{36}}$

In Exercises 29–36, simplify the expression. *(See Example 3.)*

29. $\sqrt[3]{16}$
30. $\sqrt[3]{-108}$
31. $\sqrt[3]{-64x^5}$
32. $-\sqrt[3]{343n^2}$
33. $\sqrt[3]{\frac{6c}{-125}}$
34. $\sqrt[3]{\frac{8h^4}{27}}$
35. $-\sqrt[3]{\frac{81y^2}{1000x^3}}$
36. $\sqrt[3]{\frac{21}{-64a^3b^6}}$

ERROR ANALYSIS **In Exercises 37 and 38, describe and correct the error in simplifying the expression.**

37. ✗
$$\sqrt{72} = \sqrt{4 \cdot 18} = \sqrt{4} \cdot \sqrt{18} = 2\sqrt{18}$$

38. ✗
$$\sqrt[3]{\frac{128y^3}{125}} = \frac{\sqrt[3]{128y^3}}{125} = \frac{\sqrt[3]{64 \cdot 2 \cdot y^3}}{125} = \frac{\sqrt[3]{64} \cdot \sqrt[3]{2} \cdot \sqrt[3]{y^3}}{125} = \frac{4y\sqrt[3]{2}}{125}$$

In Exercises 39–44, write a factor that you can use to rationalize the denominator of the expression.

39. $\dfrac{4}{\sqrt{6}}$

40. $\dfrac{1}{\sqrt{13z}}$

41. $\dfrac{2}{\sqrt[3]{x^2}}$

42. $\dfrac{3m}{\sqrt[3]{4}}$

43. $\dfrac{\sqrt{2}}{\sqrt{5} - 8}$

44. $\dfrac{5}{\sqrt{3} + \sqrt{7}}$

In Exercises 45–54, simplify the expression.
(See Example 4.)

45. $\dfrac{2}{\sqrt{2}}$

46. $\dfrac{4}{\sqrt{3}}$

47. $\dfrac{\sqrt{5}}{\sqrt{48}}$

48. $\sqrt{\dfrac{4}{52}}$

49. $\dfrac{3}{\sqrt{a}}$

50. $\dfrac{1}{\sqrt{2x}}$

51. $\sqrt{\dfrac{3d^2}{5}}$

52. $\dfrac{\sqrt{8}}{\sqrt{3n^3}}$

53. $\dfrac{4}{\sqrt[3]{25}}$

54. $\sqrt[3]{\dfrac{1}{108y^2}}$

In Exercises 55–60, simplify the expression.
(See Example 5.)

55. $\dfrac{1}{\sqrt{7} + 1}$

56. $\dfrac{2}{5 - \sqrt{3}}$

57. $\dfrac{\sqrt{10}}{7 - \sqrt{2}}$

58. $\dfrac{\sqrt{5}}{6 + \sqrt{5}}$

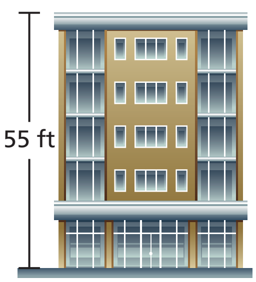
59. $\dfrac{3}{\sqrt{5} - \sqrt{2}}$

60. $\dfrac{\sqrt{3}}{\sqrt{7} + \sqrt{3}}$

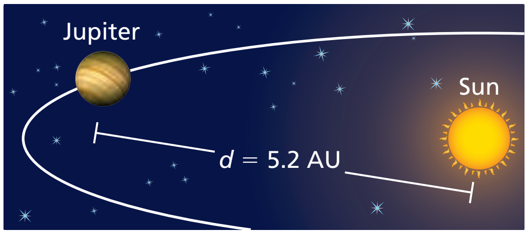
61. **MODELING WITH MATHEMATICS** The time t (in seconds) it takes an object to hit the ground is given by $t = \sqrt{\dfrac{h}{16}}$, where h is the height (in feet) from which the object was dropped. *(See Example 6.)*

a. How long does it take an earring to hit the ground when it falls from the roof of the building?

b. How much sooner does the earring hit the ground when it is dropped from two stories (22 feet) below the roof?



62. **MODELING WITH MATHEMATICS** The orbital period of a planet is the time it takes the planet to travel around the Sun. You can find the orbital period P (in Earth years) using the formula $P = \sqrt{d^3}$, where d is the average distance (in astronomical units, abbreviated AU) of the planet from the Sun.



a. Simplify the formula.

b. What is Jupiter's orbital period?

63. **MODELING WITH MATHEMATICS** The electric current I (in amperes) an appliance uses is given by the formula $I = \sqrt{\dfrac{P}{R}}$, where P is the power (in watts) and R is the resistance (in ohms). Find the current an appliance uses when the power is 147 watts and the resistance is 5 ohms.

64. **MODELING WITH MATHEMATICS** You can find the average annual interest rate r (in decimal form) of a savings account using the formula $r = \sqrt{\dfrac{V_2}{V_0}} - 1$, where V_0 is the initial investment and V_2 is the balance of the account after 2 years. Use the formula to compare the savings accounts. In which account would you invest money? Explain.

Account	Initial investment	Balance after 2 years
1	\$275	\$293
2	\$361	\$382
3	\$199	\$214
4	\$254	\$272
5	\$386	\$406

In Exercises 65–68, evaluate the function for the given value of x. Write your answer in simplest form and in decimal form rounded to the nearest hundredth.

65. $h(x) = \sqrt{5x};\ x = 10$

66. $g(x) = \sqrt{3x};\ x = 60$

67. $r(x) = \sqrt{\dfrac{3x}{3x^2 + 6}};\ x = 4$

68. $p(x) = \sqrt{\dfrac{x - 1}{5x}};\ x = 8$

In Exercises 69–72, evaluate the expression when $a = -2$, $b = 8$, and $c = \frac{1}{2}$. Write your answer in simplest form and in decimal form rounded to the nearest hundredth.

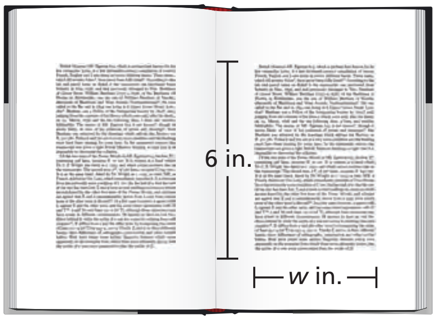
69. $\sqrt{a^2 + bc}$

70. $-\sqrt{4c - 6ab}$

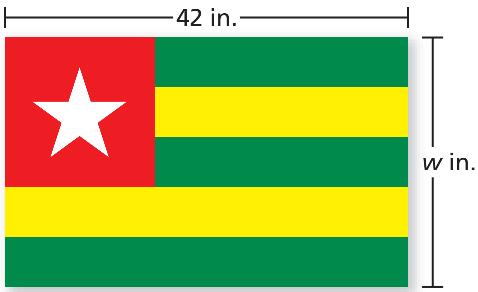
71. $-\sqrt{2a^2 + b^2}$

72. $\sqrt{b^2 - 4ac}$

73. **MODELING WITH MATHEMATICS** The text in the book shown forms a golden rectangle. What is the width w of the text? *(See Example 7.)*



74. **MODELING WITH MATHEMATICS** The flag of Togo is approximately the shape of a golden rectangle. What is the width w of the flag?



In Exercises 75–82, simplify the expression.
(See Example 8.)

75. $\sqrt{3} - 2\sqrt{2} + 6\sqrt{2}$

76. $\sqrt{5} - 5\sqrt{13} - 8\sqrt{5}$

77. $2\sqrt{6} - 5\sqrt{54}$

78. $9\sqrt{32} + \sqrt{2}$

79. $\sqrt{12} + 6\sqrt{3} + 2\sqrt{6}$

80. $3\sqrt{7} - 5\sqrt{14} + 2\sqrt{28}$

81. $\sqrt[3]{-81} + 4\sqrt[3]{3}$

82. $6\sqrt[3]{128t} - 2\sqrt[3]{2t}$

In Exercises 83–90, simplify the expression.
(See Example 9.)

83. $\sqrt{2}\left(\sqrt{45} + \sqrt{5}\right)$

84. $\sqrt{3}\left(\sqrt{72} - 3\sqrt{2}\right)$

85. $\sqrt{5}\left(2\sqrt{6x} - \sqrt{96x}\right)$

86. $\sqrt{7y}\left(\sqrt{27y} + 5\sqrt{12y}\right)$

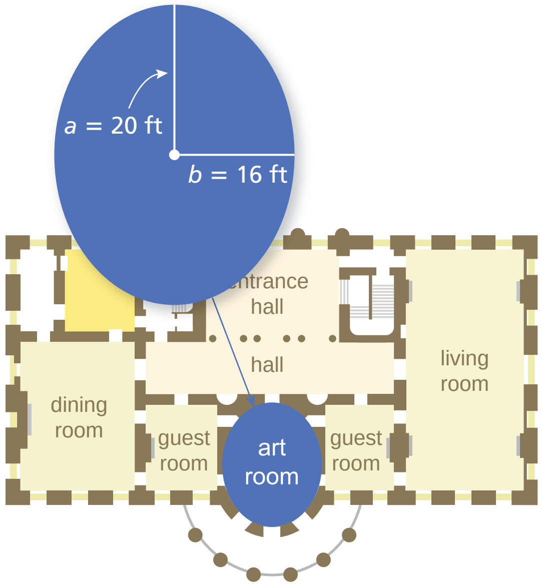
87. $\left(4\sqrt{2} - \sqrt{98}\right)^2$

88. $\left(\sqrt{3} + \sqrt{48}\right)\left(\sqrt{20} - \sqrt{5}\right)$

89. $\sqrt[3]{3}\left(\sqrt[3]{4} + \sqrt[3]{32}\right)$

90. $\sqrt[3]{2}\left(\sqrt[3]{135} - 4\sqrt[3]{5}\right)$

91. **MODELING WITH MATHEMATICS** The circumference C of the art room in a mansion is approximated by the formula $C \approx 2\pi\sqrt{\dfrac{a^2 + b^2}{2}}$. Approximate the circumference of the room.



92. **CRITICAL THINKING** Determine whether each expression represents a *rational* or an *irrational* number. Justify your answer.

a. $4 + \sqrt{6}$

b. $\dfrac{\sqrt{48}}{\sqrt{3}}$

c. $\dfrac{8}{\sqrt{12}}$

d. $\sqrt{3} + \sqrt{7}$

e. $\dfrac{a}{\sqrt{10} - \sqrt{2}}$, where a is a positive integer

f. $\dfrac{2 + \sqrt{5}}{2b + \sqrt{5b^2}}$, where b is a positive integer

In Exercises 93–98, simplify the expression.

93. $\sqrt[5]{\dfrac{13}{5x^5}}$

94. $\sqrt[4]{\dfrac{10}{81}}$

95. $\sqrt[4]{256y}$

96. $\sqrt[5]{160x^6}$

97. $6\sqrt[4]{9} - \sqrt[5]{9} + 3\sqrt[4]{9}$

98. $\sqrt[5]{2}\left(\sqrt[4]{7} + \sqrt[5]{16}\right)$

REASONING In Exercises 99 and 100, use the table shown.

	2	$\frac{1}{4}$	0	$\sqrt{3}$	$-\sqrt{3}$	π
2						
$\frac{1}{4}$						
0						
$\sqrt{3}$						
$-\sqrt{3}$						
π						

99. Copy and complete the table by (a) finding each sum $\left(2 + 2, 2 + \frac{1}{4}, \text{etc.}\right)$ and (b) finding each product $\left(2 \cdot 2, 2 \cdot \frac{1}{4}, \text{etc.}\right)$.

100. Use your answers in Exercise 99 to determine whether each statement is *always*, *sometimes*, or *never* true. Justify your answer.

a. The sum of a rational number and a rational number is rational.

b. The sum of a rational number and an irrational number is irrational.

c. The sum of an irrational number and an irrational number is irrational.

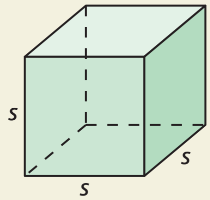
d. The product of a rational number and a rational number is rational.

e. The product of a nonzero rational number and an irrational number is irrational.

f. The product of an irrational number and an irrational number is irrational.

101. REASONING Let m be a positive integer. For what values of m will the simplified form of the expression $\sqrt{2^m}$ contain a radical? For what values will it *not* contain a radical? Explain.

102. HOW DO YOU SEE IT? The edge length s of a cube is an irrational number, the surface area is an irrational number, and the volume is a rational number. Give a possible value of s.



103. REASONING Let a and b be positive numbers. Explain why $\sqrt{ab}$ lies between a and b on a number line. (*Hint:* Let $a < b$ and multiply each side of $a < b$ by a. Then let $a < b$ and multiply each side by b.)

104. MAKING AN ARGUMENT Your friend says that you can rationalize the denominator of the expression $\frac{2}{4 + \sqrt[3]{5}}$ by multiplying the numerator and denominator by $4 - \sqrt[3]{5}$. Is your friend correct? Explain.

105. PROBLEM SOLVING The ratio of consecutive terms $\frac{a_n}{a_{n-1}}$ in the Fibonacci sequence gets closer and closer to the golden ratio $\frac{1 + \sqrt{5}}{2}$ as n increases. Find the term that precedes 610 in the sequence.

106. THOUGHT PROVOKING Use the golden ratio $\frac{1 + \sqrt{5}}{2}$ and the golden ratio conjugate $\frac{1 - \sqrt{5}}{2}$ for each of the following.

a. Show that the golden ratio and golden ratio conjugate are both solutions of $x^2 - x - 1 = 0$.

b. Construct a geometric diagram that has the golden ratio as the length of a part of the diagram.

107. CRITICAL THINKING Use the special product pattern $(a + b)(a^2 - ab + b^2) = a^3 + b^3$ to simplify the expression $\frac{2}{\sqrt[3]{x} + 1}$. Explain your reasoning.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph the linear equation. Identify the x-intercept. *(Skills Review Handbook)*

108. $y = x - 4$ **109.** $y = -2x + 6$ **110.** $y = -\frac{1}{3}x - 1$ **111.** $y = \frac{3}{2}x + 6$

Find the product. *(Section 2.3)*

112. $(z + 3)^2$ **113.** $(3a - 5b)^2$ **114.** $(x + 8)(x - 8)$ **115.** $(4y + 2)(4y - 2)$

4.2 Solving Quadratic Equations by Graphing

Essential Question

Essential Question How can you use a graph to solve a quadratic equation in one variable?

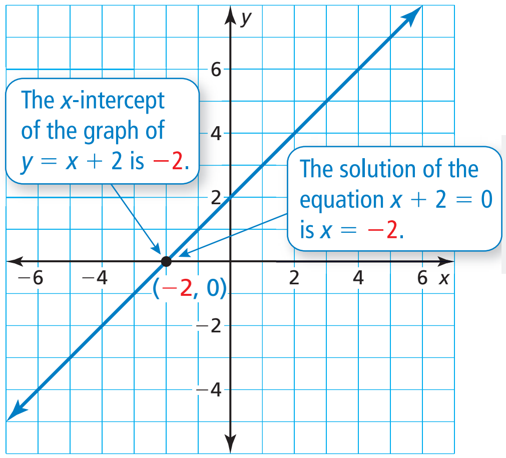
Based on the definition of an x-intercept of a graph, it follows that the x-intercept of the graph of the linear equation

$y = ax + b$ 2 variables

is the same value as the solution of

$ax + b = 0.$ 1 variable

You can use similar reasoning to solve *quadratic equations*.



EXPLORATION 1 Solving a Quadratic Equation by Graphing

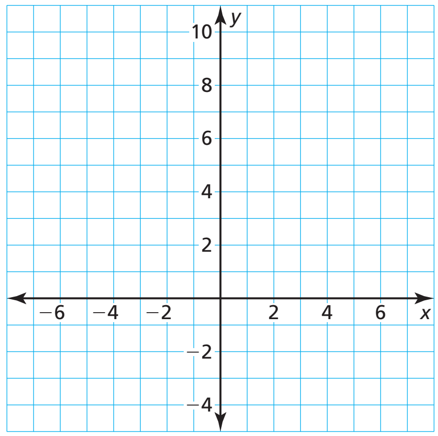
Work with a partner.

a. Sketch the graph of $y = x^2 - 2x$.

b. What is the definition of an x-intercept of a graph? How many x-intercepts does this graph have? What are they?

c. What is the definition of a solution of an equation in x? How many solutions does the equation $x^2 - 2x = 0$ have? What are they?

d. Explain how you can verify the solutions you found in part (c).



EXPLORATION 2 Solving Quadratic Equations by Graphing

Work with a partner. Solve each equation by graphing.

a. $x^2 - 4 = 0$ **b.** $x^2 + 3x = 0$

c. $-x^2 + 2x = 0$ **d.** $x^2 - 2x + 1 = 0$

e. $x^2 - 3x + 5 = 0$ **f.** $-x^2 + 3x - 6 = 0$

MAKING SENSE OF PROBLEMS

To be proficient in math, you need to check your answers to problems using a different method and continually ask yourself, "Does this make sense?"

Communicate Your Answer

3. How can you use a graph to solve a quadratic equation in one variable?

4. After you find a solution graphically, how can you check your result algebraically? Check your solutions for parts (a)–(d) in Exploration 2 algebraically.

5. How can you determine graphically that a quadratic equation has no solution?

4.2 Lesson

Core Vocabulary

quadratic equation, *p. 202*

Previous
x-intercept
root
zero of a function

What You Will Learn

- Solve quadratic equations by graphing.
- Use graphs to find and approximate the zeros of functions.
- Solve real-life problems using graphs of quadratic functions.

Solving Quadratic Equations by Graphing

A **quadratic equation** is a nonlinear equation that can be written in the standard form $ax^2 + bx + c = 0$, where $a \neq 0$.

In Chapter 2, you solved quadratic equations by factoring. You can also solve quadratic equations by graphing.

Core Concept

Solving Quadratic Equations by Graphing

Step 1 Write the equation in standard form, $ax^2 + bx + c = 0$.

Step 2 Graph the related function $y = ax^2 + bx + c$.

Step 3 Find the x-intercepts, if any.

The solutions, or *roots*, of $ax^2 + bx + c = 0$ are the x-intercepts of the graph.

EXAMPLE 1 Solving a Quadratic Equation: Two Real Solutions

Solve $x^2 + 2x = 3$ by graphing.

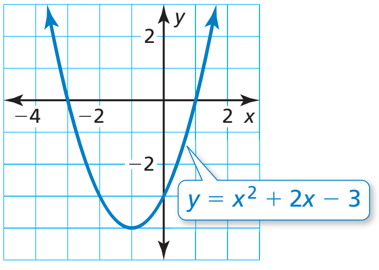
SOLUTION

Step 1 Write the equation in standard form.

$$x^2 + 2x = 3 \qquad \text{Write original equation.}$$

$$x^2 + 2x - 3 = 0 \qquad \text{Subtract 3 from each side.}$$

Step 2 Graph the related function $y = x^2 + 2x - 3$.



Step 3 Find the x-intercepts. The x-intercepts are -3 and 1.

- So, the solutions are $x = -3$ and $x = 1$.

Check

$x^2 + 2x = 3$	Original equation	$x^2 + 2x = 3$
$(-3)^2 + 2(-3) \stackrel{?}{=} 3$	Substitute.	$1^2 + 2(1) \stackrel{?}{=} 3$
$3 = 3$ ✓	Simplify.	$3 = 3$ ✓

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Solve the equation by graphing. Check your solutions.

1. $x^2 - x - 2 = 0$ **2.** $x^2 + 7x = -10$ **3.** $x^2 + x = 12$

EXAMPLE 2 Solving a Quadratic Equation: One Real Solution

Solve $x^2 - 8x = -16$ by graphing.

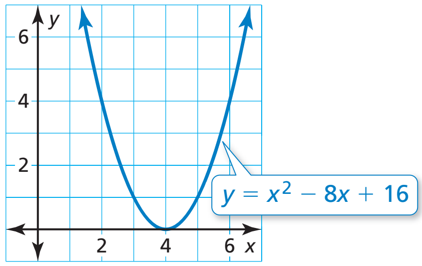
SOLUTION

Step 1 Write the equation in standard form.

$$x^2 - 8x = -16 \qquad \text{Write original equation.}$$

$$x^2 - 8x + 16 = 0 \qquad \text{Add 16 to each side.}$$

Step 2 Graph the related function $y = x^2 - 8x + 16$.



Step 3 Find the x-intercept. The only x-intercept is at the vertex, $(4, 0)$.

▶ So, the solution is $x = 4$.

ANOTHER WAY

You can also solve the equation in Example 2 by factoring.

$$x^2 - 8x + 16 = 0$$

$$(x - 4)(x - 4) = 0$$

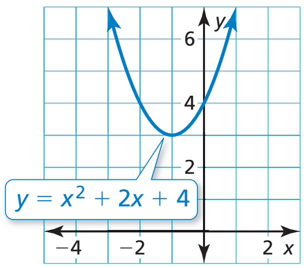
So, $x = 4$.

EXAMPLE 3 Solving a Quadratic Equation: No Real Solutions

Solve $-x^2 = 2x + 4$ by graphing.

SOLUTION

Method 1 Write the equation in standard form, $x^2 + 2x + 4 = 0$. Then graph the related function $y = x^2 + 2x + 4$, as shown at the left.

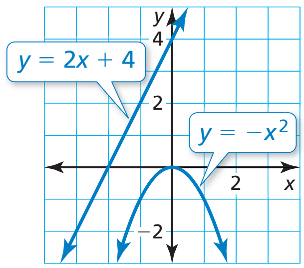


▶ There are no x-intercepts. So, $-x^2 = 2x + 4$ has no real solutions.

Method 2 Graph each side of the equation.

$$y = -x^2 \qquad \text{Left side}$$

$$y = 2x + 4 \qquad \text{Right side}$$



▶ The graphs do not intersect. So, $-x^2 = 2x + 4$ has no real solutions.

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Solve the equation by graphing.

4. $x^2 + 36 = 12x$ **5.** $x^2 + 4x = 0$ **6.** $x^2 + 10x = -25$

7. $x^2 = 3x - 3$ **8.** $x^2 + 7x = -6$ **9.** $2x + 5 = -x^2$

Concept Summary

Number of Solutions of a Quadratic Equation

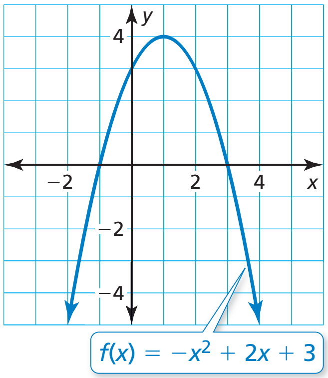
A quadratic equation has:

- two real solutions when the graph of its related function has two x-intercepts.
- one real solution when the graph of its related function has one x-intercept.
- no real solutions when the graph of its related function has no x-intercepts.

Finding Zeros of Functions

Recall that a zero of a function is an x-intercept of the graph of the function.

EXAMPLE 4 Finding the Zeros of a Function



The graph of $f(x) = -x^2 + 2x + 3$ is shown. Find the zeros of f.

SOLUTION

The x-intercepts are -1 and 3.

▶ So, the zeros of f are -1 and 3.

Check

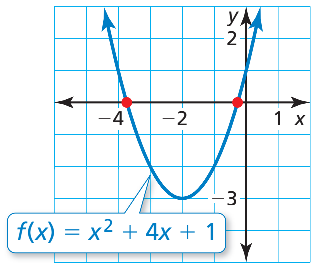
$f(-1) = -(-1)^2 + 2(-1) + 3 = 0$ ✓

$f(3) = -3^2 + 2(3) + 3 = 0$ ✓

The zeros of a function are not necessarily integers. To approximate zeros, analyze the signs of function values. When two function values have different signs, a zero lies between the x-values that correspond to the function values.

EXAMPLE 5 Approximating the Zeros of a Function

The graph of $f(x) = x^2 + 4x + 1$ is shown. Approximate the zeros of f to the nearest tenth.



SOLUTION

There are two x-intercepts: one between -4 and -3, and another between -1 and 0.

Make tables using x-values between -4 and -3, and between -1 and 0. Use an increment of 0.1. Look for a change in the signs of the function values.

x	-3.9	-3.8	-3.7	-3.6	-3.5	-3.4	-3.3	-3.2	-3.1
$f(x)$	0.61	0.24	-0.11	-0.44	-0.75	-1.04	-1.31	-1.56	-1.79

change in signs

x	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
$f(x)$	-1.79	-1.56	-1.31	-1.04	-0.75	-0.44	-0.11	0.24	0.61

change in signs

ANOTHER WAY

You could approximate one zero using a table and then use the axis of symmetry to find the other zero.

The function values that are closest to 0 correspond to x-values that best approximate the zeros of the function.

▶ In each table, the function value closest to 0 is -0.11. So, the zeros of f are about -3.7 and -0.3.

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10. Graph $f(x) = x^2 + x - 6$. Find the zeros of f.

11. Graph $f(x) = -x^2 + 2x + 2$. Approximate the zeros of f to the nearest tenth.

Solving Real-Life Problems

EXAMPLE 6 **Real-Life Application**

A football player kicks a football 2 feet above the ground with an initial vertical velocity of 75 feet per second. The function $h = -16t^2 + 75t + 2$ represents the height h (in feet) of the football after t seconds. (a) Find the height of the football each second after it is kicked. (b) Use the results of part (a) to estimate when the height of the football is 50 feet. (c) Using a graph, after how many seconds is the football 50 feet above the ground?

SOLUTION

a. Make a table of values starting with $t = 0$ seconds using an increment of 1. Continue the table until a function value is negative.

Seconds, t	Height, h
0	2
1	61
2	88
3	83
4	46
5	-23

▶ The height of the football is 61 feet after 1 second, 88 feet after 2 seconds, 83 feet after 3 seconds, and 46 feet after 4 seconds.

b. From part (a), you can estimate that the height of the football is 50 feet between 0 and 1 second and between 3 and 4 seconds.

▶ Based on the function values, it is reasonable to estimate that the height of the football is 50 feet slightly less than 1 second and slightly less than 4 seconds after it is kicked.

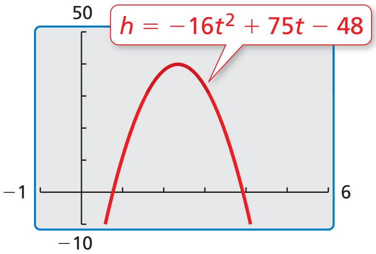
c. To determine when the football is 50 feet above the ground, find the t-values for which $h = 50$. So, solve the equation $-16t^2 + 75t + 2 = 50$ by graphing.

Step 1 Write the equation in standard form.

$$-16t^2 + 75t + 2 = 50 \qquad \text{Write the equation.}$$

$$-16t^2 + 75t - 48 = 0 \qquad \text{Subtract 50 from each side.}$$

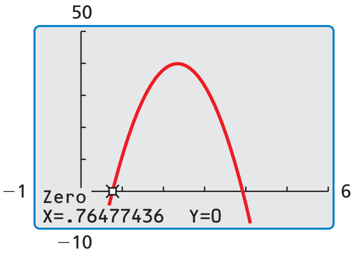
Step 2 Use a graphing calculator to graph the related function $h = -16t^2 + 75t - 48$.

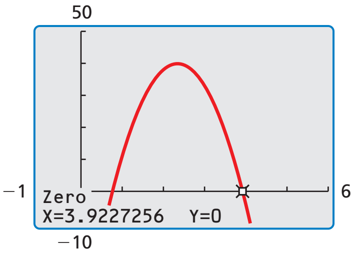


REMEMBER

Equations have *solutions*, or *roots*. Graphs have *x-intercepts*. Functions have *zeros*.

Step 3 Use the *zero* feature to find the zeros of the function.





▶ The football is 50 feet above the ground after about 0.8 second and about 3.9 seconds, which supports the estimates in part (b).

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12. WHAT IF? After how many seconds is the football 65 feet above the ground?

4.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What is a quadratic equation?

2. **WHICH ONE DOESN'T BELONG?** Which equation does *not* belong with the other three? Explain your reasoning.

$x^2 + 5x = 20$	$x^2 + x - 4 = 0$	$x^2 - 6 = 4x$	$7x + 12 = x^2$

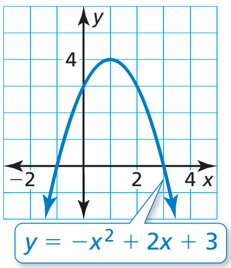
3. **WRITING** How can you use a graph to find the number of solutions of a quadratic equation?

4. **WRITING** How are solutions, roots, x-intercepts, and zeros related?

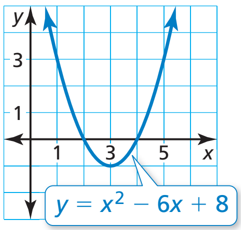
Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, use the graph to solve the equation.

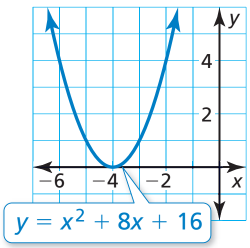
5. $-x^2 + 2x + 3 = 0$



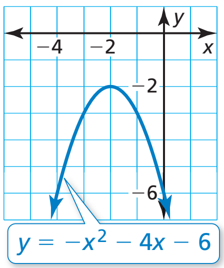
6. $x^2 - 6x + 8 = 0$



7. $x^2 + 8x + 16 = 0$



8. $-x^2 - 4x - 6 = 0$



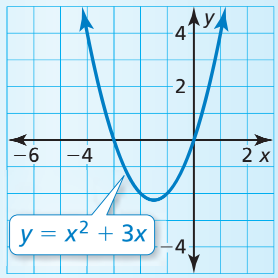
In Exercises 9–12, write the equation in standard form.

9. $4x^2 = 12$
10. $-x^2 = 15$
11. $2x - x^2 = 1$
12. $5 + x = 3x^2$

In Exercises 13–24, solve the equation by graphing. *(See Examples 1, 2, and 3.)*

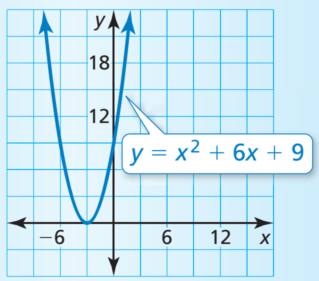
13. $x^2 - 5x = 0$
14. $x^2 - 4x + 4 = 0$
15. $x^2 - 2x + 5 = 0$
16. $x^2 - 6x - 7 = 0$
17. $x^2 = 6x - 9$
18. $-x^2 = 8x + 20$
19. $x^2 = -1 - 2x$
20. $x^2 = -x - 3$
21. $4x - 12 = -x^2$
22. $5x - 6 = x^2$
23. $x^2 - 2 = -x$
24. $16 + x^2 = -8x$

25. **ERROR ANALYSIS** Describe and correct the error in solving $x^2 + 3x = 18$ by graphing.



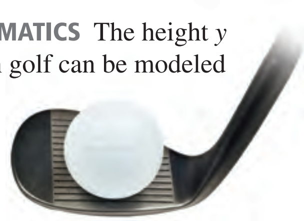
The solutions of the equation $x^2 + 3x = 18$ are $x = -3$ and $x = 0$.

26. **ERROR ANALYSIS** Describe and correct the error in solving $x^2 + 6x + 9 = 0$ by graphing.



The solution of the equation $x^2 + 6x + 9 = 0$ is $x = 9$.

27. **MODELING WITH MATHEMATICS** The height y (in yards) of a flop shot in golf can be modeled by $y = -x^2 + 5x$, where x is the horizontal distance (in yards).



a. Interpret the x-intercepts of the graph of the equation.

b. How far away does the golf ball land?

28. **MODELING WITH MATHEMATICS** The height h (in feet) of an underhand volleyball serve can be modeled by $h = -16t^2 + 30t + 4$, where t is the time (in seconds).

a. Do both t-intercepts of the graph of the function have meaning in this situation? Explain.

b. No one receives the serve. After how many seconds does the volleyball hit the ground?

In Exercises 29–36, solve the equation by using Method 2 from Example 3.

29. $x^2 = 10 - 3x$

30. $2x - 3 = x^2$

31. $5x - 7 = x^2$

32. $x^2 = 6x - 5$

33. $x^2 + 12x = -20$

34. $x^2 + 8x = 9$

35. $-x^2 - 5 = -2x$

36. $-x^2 - 4 = -4x$

In Exercises 37–42, find the zero(s) of *f*. *(See Example 4.)*

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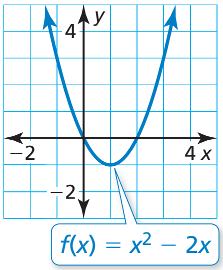

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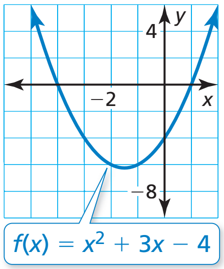

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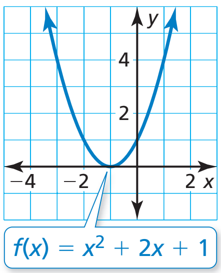

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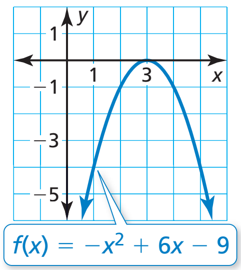

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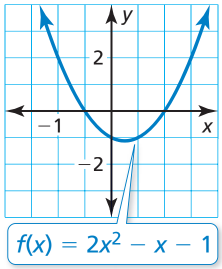

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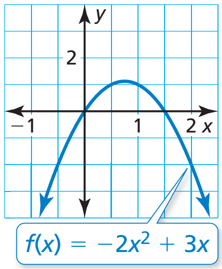

In Exercises 43–46, approximate the zeros of *f* to the nearest tenth. *(See Example 5.)*

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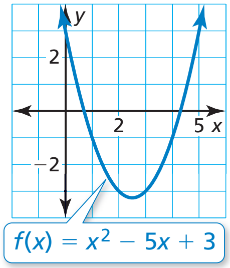

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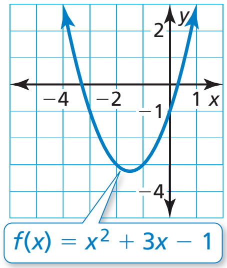

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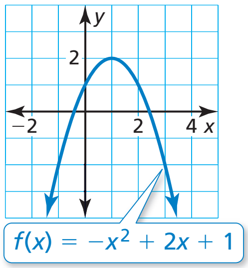

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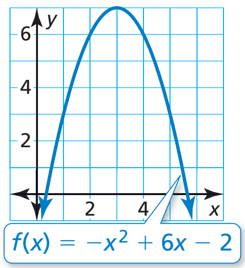


In Exercises 47–52, graph the function. Approximate the zeros of the function to the nearest tenth, if necessary.

47. $f(x) = x^2 + 6x + 1$

48. $f(x) = x^2 - 3x + 2$

49. $y = -x^2 + 4x - 2$

50. $y = -x^2 + 9x - 6$

51. $f(x) = \frac{1}{2}x^2 + 2x - 5$

52. $f(x) = -3x^2 + 4x + 3$

53. **MODELING WITH MATHEMATICS** At a Civil War reenactment, a cannonball is fired into the air with an initial vertical velocity of 128 feet per second. The release point is 6 feet above the ground. The function $h = -16t^2 + 128t + 6$ represents the height h (in feet) of the cannonball after t seconds. *(See Example 6.)*

a. Find the height of the cannonball each second after it is fired.

b. Use the results of part (a) to estimate when the height of the cannonball is 150 feet.

c. Using a graph, after how many seconds is the cannonball 150 feet above the ground?

54. **MODELING WITH MATHEMATICS** You throw a softball straight up into the air with an initial vertical velocity of 40 feet per second. The release point is 5 feet above the ground. The function $h = -16t^2 + 40t + 5$ represents the height h (in feet) of the softball after t seconds.

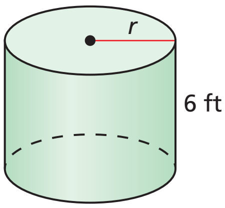
a. Find the height of the softball each second after it is released.

b. Use the results of part (a) to estimate when the height of the softball is 15 feet.

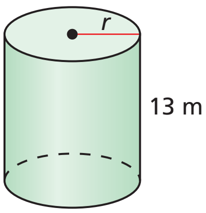
c. Using a graph, after how many seconds is the softball 15 feet above the ground?

MATHEMATICAL CONNECTIONS **In Exercises 55 and 56, use the given surface area S of the cylinder to find the radius r to the nearest tenth.**

55. $S = 225\text{ ft}^2$

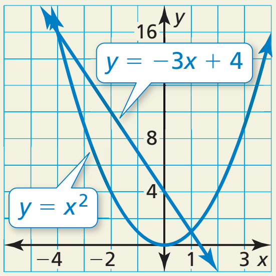


56. $S = 750\text{ m}^2$



57. **WRITING** Explain how to approximate zeros of a function when the zeros are not integers.

58. **HOW DO YOU SEE IT?** Consider the graph shown.



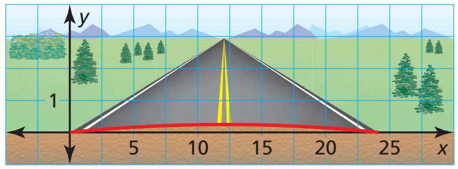
a. How many solutions does the quadratic equation $x^2 = -3x + 4$ have? Explain.

b. Without graphing, describe what you know about the graph of $y = x^2 + 3x - 4$.

59. **COMPARING METHODS** Example 3 shows two methods for solving a quadratic equation. Which method do you prefer? Explain your reasoning.

60. **THOUGHT PROVOKING** How many different parabolas have -2 and 2 as x-intercepts? Sketch examples of parabolas that have these two x-intercepts.

61. **MODELING WITH MATHEMATICS** To keep water off a road, the surface of the road is shaped like a parabola. A cross section of the road is shown in the diagram. The surface of the road can be modeled by $y = -0.0017x^2 + 0.041x$, where x and y are measured in feet. Find the width of the road to the nearest tenth of a foot.



62. **MAKING AN ARGUMENT** A stream of water from a fire hose can be modeled by $y = -0.003x^2 + 0.58x + 3$, where x and y are measured in feet. A firefighter is standing 57 feet from a building and is holding the hose 3 feet above the ground. The bottom of a window of the building is 26 feet above the ground. Your friend claims the stream of water will pass through the window. Is your friend correct? Explain.

REASONING **In Exercises 63–65, determine whether the statement is *always*, *sometimes*, or *never* true. Justify your answer.**

63. The graph of $y = ax^2 + c$ has two x-intercepts when a is negative.

64. The graph of $y = ax^2 + c$ has no x-intercepts when a and c have the same sign.

65. The graph of $y = ax^2 + bx + c$ has more than two x-intercepts when $a \neq 0$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Show that an exponential model fits the data. Then write a recursive rule that models the data. *(Section 1.6)*

66.

n	0	1	2	3
$f(n)$	18	3	$\frac{1}{2}$	$\frac{1}{12}$

67.

n	0	1	2	3
$f(n)$	2	8	32	128

4.3 Solving Quadratic Equations Using Square Roots

Essential Question

How can you determine the number of solutions of a quadratic equation of the form $ax^2 + c = 0$?

EXPLORATION 1 The Number of Solutions of $ax^2 + c = 0$

Work with a partner. Solve each equation by graphing. Explain how the number of solutions of $ax^2 + c = 0$ relates to the graph of $y = ax^2 + c$.

a. $x^2 - 4 = 0$

b. $2x^2 + 5 = 0$

c. $x^2 = 0$

d. $x^2 - 5 = 0$

EXPLORATION 2 Estimating Solutions

Work with a partner. Complete each table. Use the completed tables to estimate the solutions of $x^2 - 5 = 0$. Explain your reasoning.

a.

x	$x^2 - 5$
2.21	
2.22	
2.23	
2.24	
2.25	
2.26	

b.

x	$x^2 - 5$
-2.21	
-2.22	
-2.23	
-2.24	
-2.25	
-2.26	

EXPLORATION 3 Using Technology to Estimate Solutions

Work with a partner. Two equations are equivalent when they have the same solutions.

a. Are the equations $x^2 - 5 = 0$ and $x^2 = 5$ equivalent? Explain your reasoning.

b. Use the square root key on a calculator to estimate the solutions of $x^2 - 5 = 0$. Describe the accuracy of your estimates in Exploration 2.

c. Write the exact solutions of $x^2 - 5 = 0$.

ATTENDING TO PRECISION

To be proficient in math, you need to calculate accurately and express numerical answers with a level of precision appropriate for the problem's context.

Communicate Your Answer

4. How can you determine the number of solutions of a quadratic equation of the form $ax^2 + c = 0$?

5. Write the exact solutions of each equation. Then use a calculator to estimate the solutions.

a. $x^2 - 2 = 0$

b. $3x^2 - 18 = 0$

c. $x^2 = 8$

4.3 Lesson

Core Vocabulary

Previous
square root
zero of a function

What You Will Learn

- Solve quadratic equations using square roots.
- Approximate the solutions of quadratic equations.

Solving Quadratic Equations Using Square Roots

Earlier in this chapter, you studied properties of square roots. Now you will use square roots to solve quadratic equations of the form $ax^2 + c = 0$. First isolate x^2 on one side of the equation to obtain $x^2 = d$. Then solve by taking the square root of each side.

Core Concept

Solutions of $x^2 = d$

- When $d > 0$, $x^2 = d$ has two real solutions, $x = \pm\sqrt{d}$.
- When $d = 0$, $x^2 = d$ has one real solution, $x = 0$.
- When $d < 0$, $x^2 = d$ has no real solutions.

ANOTHER WAY

You can also solve $3x^2 - 27 = 0$ by factoring.

$$3(x^2 - 9) = 0$$
$$3(x - 3)(x + 3) = 0$$
$$x = 3 \text{ or } x = -3$$

EXAMPLE 1 Solving Quadratic Equations Using Square Roots

a. Solve $3x^2 - 27 = 0$ using square roots.

$3x^2 - 27 = 0$	Write the equation.
$3x^2 = 27$	Add 27 to each side.
$x^2 = 9$	Divide each side by 3.
$x = \pm\sqrt{9}$	Take the square root of each side.
$x = \pm 3$	Simplify.

▶ The solutions are $x = 3$ and $x = -3$.

b. Solve $x^2 - 10 = -10$ using square roots.

$x^2 - 10 = -10$	Write the equation.
$x^2 = 0$	Add 10 to each side.
$x = 0$	Take the square root of each side.

▶ The only solution is $x = 0$.

c. Solve $-5x^2 + 11 = 16$ using square roots.

$-5x^2 + 11 = 16$	Write the equation.
$-5x^2 = 5$	Subtract 11 from each side.
$x^2 = -1$	Divide each side by -5.

▶ The square of a real number cannot be negative. So, the equation has no real solutions.

STUDY TIP

Each side of the equation $(x-1)^2 = 25$ is a square. So, you can still solve by taking the square root of each side.

EXAMPLE 2 Solving a Quadratic Equation Using Square Roots

Solve $(x-1)^2 = 25$ using square roots.

SOLUTION

$(x-1)^2 = 25$ — Write the equation.

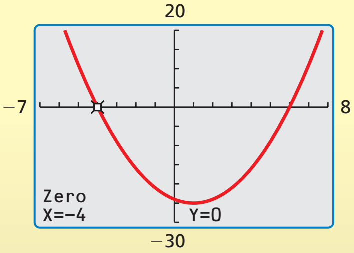
$x - 1 = \pm 5$ — Take the square root of each side.

$x = 1 \pm 5$ — Add 1 to each side.

▶ So, the solutions are $x = 1 + 5 = 6$ and $x = 1 - 5 = -4$.

Check

Use a graphing calculator to check your answer. Rewrite the equation as $(x-1)^2 - 25 = 0$. Graph the related function $f(x) = (x-1)^2 - 25$ and find the zeros of the function. The zeros are -4 and 6.



Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Solve the equation using square roots.

1. $-3x^2 = -75$ **2.** $x^2 + 12 = 10$ **3.** $4x^2 - 15 = -15$

4. $(x+7)^2 = 0$ **5.** $4(x-3)^2 = 9$ **6.** $(2x+1)^2 = 36$

Approximating Solutions of Quadratic Equations

EXAMPLE 3 Approximating Solutions of a Quadratic Equation

Solve $4x^2 - 13 = 15$ using square roots. Round the solutions to the nearest hundredth.

SOLUTION

$4x^2 - 13 = 15$ — Write the equation.

$4x^2 = 28$ — Add 13 to each side.

$x^2 = 7$ — Divide each side by 4.

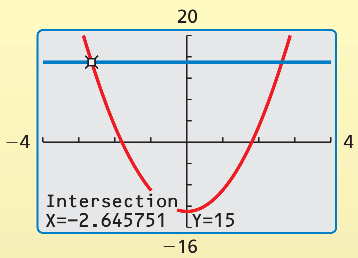
$x = \pm\sqrt{7}$ — Take the square root of each side.

$x \approx \pm 2.65$ — Use a calculator.

▶ The solutions are $x \approx -2.65$ and $x \approx 2.65$.

Check

Graph each side of the equation and find the points of intersection. The x-values of the points of intersection are about -2.65 and 2.65.



Monitoring Progress

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Solve the equation using square roots. Round your solutions to the nearest hundredth.

7. $x^2 + 8 = 19$ **8.** $5x^2 - 2 = 0$ **9.** $3x^2 - 30 = 4$

EXAMPLE 4 Solving a Real-Life Problem

A touch tank has a height of 3 feet. Its length is three times its width. The volume of the tank is 270 cubic feet. Find the length and width of the tank.

SOLUTION

The length ℓ is three times the width w, so $\ell = 3w$. Write an equation using the formula for the volume of a rectangular prism.

$V = \ell wh$ Write the formula.

$270 = 3w(w)(3)$ Substitute 270 for V, $3w$ for ℓ, and 3 for h.

$270 = 9w^2$ Multiply.

$30 = w^2$ Divide each side by 9.

$\pm\sqrt{30} = w$ Take the square root of each side.

The solutions are $\sqrt{30}$ and $-\sqrt{30}$. Use the positive solution.

▶ So, the width is $\sqrt{30} \approx 5.5$ feet and the length is $3\sqrt{30} \approx 16.4$ feet.

INTERPRETING MATHEMATICAL RESULTS

Use the positive square root because negative solutions do not make sense in this context. Length and width cannot be negative.

EXAMPLE 5 Rearranging and Evaluating a Formula

The area A of an equilateral triangle with side length s is given by the formula $A = \frac{\sqrt{3}}{4}s^2$. Solve the formula for s. Then approximate the side length of the traffic sign that has an area of 390 square inches.

SOLUTION

Step 1 Solve the formula for s.

$A = \frac{\sqrt{3}}{4}s^2$ Write the formula.

$\frac{4A}{\sqrt{3}} = s^2$ Multiply each side by $\frac{4}{\sqrt{3}}$.

$\sqrt{\frac{4A}{\sqrt{3}}} = s$ Take the positive square root of each side.

Step 2 Substitute 390 for A in the new formula and evaluate.

$s = \sqrt{\frac{4A}{\sqrt{3}}} = \sqrt{\frac{4(390)}{\sqrt{3}}} = \sqrt{\frac{1560}{\sqrt{3}}} \approx 30$ Use a calculator.

▶ The side length of the traffic sign is about 30 inches.

ANOTHER WAY

Notice that you can rewrite the formula as $s = \frac{2}{3^{1/4}}\sqrt{A}$, or $s \approx 1.52\sqrt{A}$. This can help you efficiently find the value of s for various values of A.

Monitoring Progress

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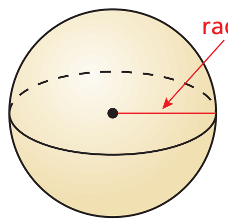
10. **WHAT IF?** In Example 4, the volume of the tank is 315 cubic feet. Find the length and width of the tank.

11. The surface area S of a sphere with radius r is given by the formula $S = 4\pi r^2$. Solve the formula for r. Then find the radius of a globe with a surface area of 804 square inches.



4.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The equation $x^2 = d$ has ____ real solutions when $d > 0$.

2. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

Solve $x^2 = 144$ using square roots.	Solve $x^2 - 144 = 0$ using square roots.
Solve $x^2 + 146 = 2$ using square roots.	Solve $x^2 + 2 = 146$ using square roots.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, determine the number of real solutions of the equation. Then solve the equation using square roots.

3. $x^2 = 25$
4. $x^2 = -36$
5. $x^2 = -21$
6. $x^2 = 400$
7. $x^2 = 0$
8. $x^2 = 169$

In Exercises 9–18, solve the equation using square roots. *(See Example 1.)*

9. $x^2 - 16 = 0$
10. $x^2 + 6 = 0$
11. $3x^2 + 12 = 0$
12. $x^2 - 55 = 26$
13. $2x^2 - 98 = 0$
14. $-x^2 + 9 = 9$
15. $-3x^2 - 5 = -5$
16. $4x^2 - 371 = 29$
17. $4x^2 + 10 = 11$
18. $9x^2 - 35 = 14$

In Exercises 19–24, solve the equation using square roots. *(See Example 2.)*

19. $(x + 3)^2 = 0$
20. $(x - 1)^2 = 4$
21. $(2x - 1)^2 = 81$
22. $(4x + 5)^2 = 9$
23. $9(x + 1)^2 = 16$
24. $4(x - 2)^2 = 25$

In Exercises 25–30, solve the equation using square roots. Round your solutions to the nearest hundredth. *(See Example 3.)*

25. $x^2 + 6 = 13$
26. $x^2 + 11 = 24$
27. $2x^2 - 9 = 11$
28. $5x^2 + 2 = 6$
29. $-21 = 15 - 2x^2$
30. $2 = 4x^2 - 5$

31. **ERROR ANALYSIS** Describe and correct the error in solving the equation $2x^2 - 33 = 39$ using square roots.

✗

$$2x^2 - 33 = 39$$
$$2x^2 = 72$$
$$x^2 = 36$$
$$x = 6$$

The solution is $x = 6$.

32. **MODELING WITH MATHEMATICS** An in-ground pond has the shape of a rectangular prism. The pond has a depth of 24 inches and a volume of 72,000 cubic inches. The length of the pond is two times its width. Find the length and width of the pond. *(See Example 4.)*

33. **MODELING WITH MATHEMATICS** A person sitting in the top row of the bleachers at a sporting event drops a pair of sunglasses from a height of 24 feet. The function $h = -16x^2 + 24$ represents the height h (in feet) of the sunglasses after x seconds. How long does it take the sunglasses to hit the ground?

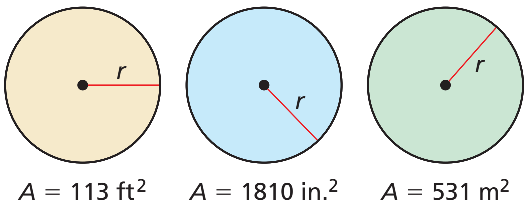
34. **MAKING AN ARGUMENT** Your friend says that the solution of the equation $x^2 + 4 = 0$ is $x = 0$. Your cousin says that the equation has no real solutions. Who is correct? Explain your reasoning.

35. **MODELING WITH MATHEMATICS** The design of a square rug for your living room is shown. You want the area of the inner square to be 25% of the total area of the rug. Find the side length x of the inner square.

36. **MATHEMATICAL CONNECTIONS** The area A of a circle with radius r is given by the formula $A = \pi r^2$. *(See Example 5.)*

a. Solve the formula for r.

b. Use the formula from part (a) to find the radius of each circle.



c. Explain why it is beneficial to solve the formula for r before finding the radius.

37. **WRITING** How can you approximate the roots of a quadratic equation when the roots are not integers?

38. **WRITING** Given the equation $ax^2 + c = 0$, describe the values of a and c so the equation has the following number of solutions.

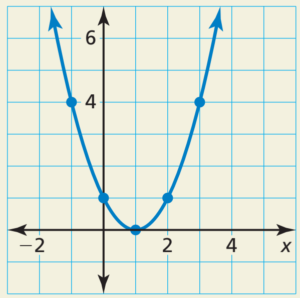
a. two real solutions

b. one real solution

c. no real solutions

39. **REASONING** Without graphing, where do the graphs of $y = x^2$ and $y = 9$ intersect? Explain.

40. **HOW DO YOU SEE IT?** The graph represents the function $f(x) = (x - 1)^2$. How many solutions does the equation $(x - 1)^2 = 0$ have? Explain.



41. **REASONING** Solve $x^2 = 1.44$ without using a calculator. Explain your reasoning.

42. **THOUGHT PROVOKING** The quadratic equation

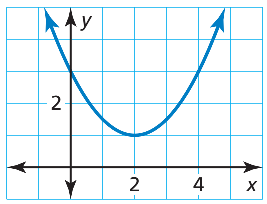
$$ax^2 + bx + c = 0$$

can be rewritten in the following form.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Use this form to write the solutions of the equation.

43. **REASONING** An equation of the graph shown is $y = \frac{1}{2}(x - 2)^2 + 1$. Two points on the parabola have y-coordinates of 9. Find the x-coordinates of these points.



44. **CRITICAL THINKING** Solve each equation without graphing.

a. $x^2 - 12x + 36 = 64$

b. $x^2 + 14x + 49 = 16$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Factor the polynomial. *(Section 2.7)*

45. $x^2 + 8x + 16$

46. $x^2 - 4x + 4$

47. $x^2 - 14x + 49$

48. $x^2 + 18x + 81$

49. $x^2 + 12x + 36$

50. $x^2 - 22x + 121$

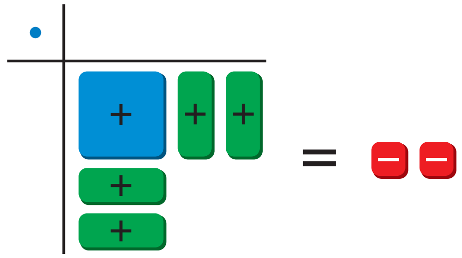
4.4 Solving Quadratic Equations by Completing the Square

Essential Question How can you use “completing the square” to solve a quadratic equation?

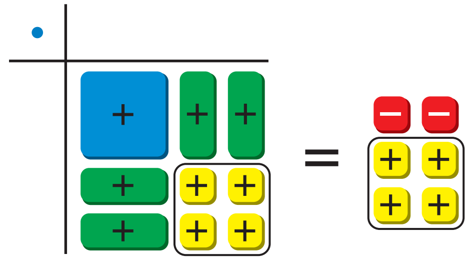
EXPLORATION 1 Solving by Completing the Square

Work with a partner.

a. Write the equation modeled by the algebra tiles. This is the equation to be solved.

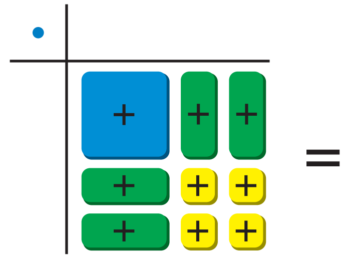


b. Four algebra tiles are added to the left side to “complete the square.” Why are four algebra tiles also added to the right side?



c. Use algebra tiles to label the dimensions of the square on the left side and simplify on the right side.

d. Write the equation modeled by the algebra tiles so that the left side is the square of a binomial. Solve the equation using square roots.



MAKING SENSE OF PROBLEMS

To be proficient in math, you need to explain to yourself the meaning of a problem. After that, you need to look for entry points to its solution.

EXPLORATION 2 Solving by Completing the Square

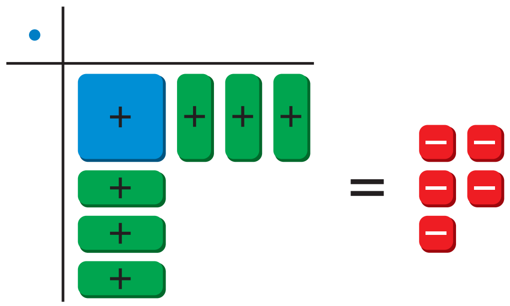
Work with a partner.

a. Write the equation modeled by the algebra tiles.

b. Use algebra tiles to “complete the square.”

c. Write the solutions of the equation.

d. Check each solution in the original equation.



Communicate Your Answer

3. How can you use “completing the square” to solve a quadratic equation?

4. Solve each quadratic equation by completing the square.

a. $x^2 - 2x = 1$ **b.** $x^2 - 4x = -1$ **c.** $x^2 + 4x = -3$

4.4 Lesson

Core Vocabulary

completing the square, *p. 216*

Previous
perfect square trinomial
coefficient
maximum value
minimum value
vertex form of a quadratic function

What You Will Learn

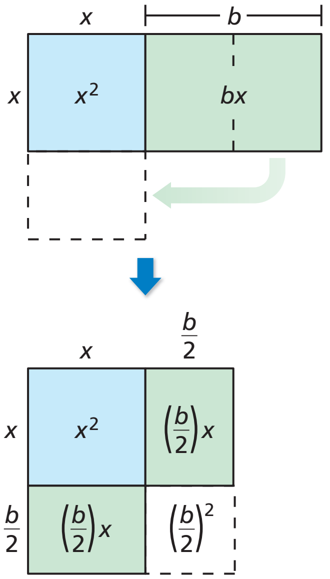
- Complete the square for expressions of the form $x^2 + bx$.
- Solve quadratic equations by completing the square.
- Find and use maximum and minimum values.
- Solve real-life problems by completing the square.

Completing the Square

For an expression of the form $x^2 + bx$, you can add a constant c to the expression so that $x^2 + bx + c$ is a perfect square trinomial. This process is called **completing the square**.

JUSTIFYING STEPS

In each diagram below, the combined area of the shaded regions is $x^2 + bx$. Adding $\left(\frac{b}{2}\right)^2$ completes the square in the second diagram.



Core Concept

Completing the Square

Words To complete the square for an expression of the form $x^2 + bx$, follow these steps.

Step 1 Find one-half of b, the coefficient of x.

Step 2 Square the result from Step 1.

Step 3 Add the result from Step 2 to $x^2 + bx$.

Factor the resulting expression as the square of a binomial.

Algebra $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

EXAMPLE 1 Completing the Square

Complete the square for each expression. Then factor the trinomial.

a. $x^2 + 6x$ **b.** $x^2 - 9x$

SOLUTION

a. Step 1 Find one-half of b. $\frac{b}{2} = \frac{6}{2} = 3$

Step 2 Square the result from Step 1. $3^2 = 9$

Step 3 Add the result from Step 2 to $x^2 + bx$. $x^2 + 6x + 9$

$x^2 + 6x + 9 = (x + 3)^2$

b. Step 1 Find one-half of b. $\frac{b}{2} = \frac{-9}{2}$

Step 2 Square the result from Step 1. $\left(\frac{-9}{2}\right)^2 = \frac{81}{4}$

Step 3 Add the result from Step 2 to $x^2 + bx$. $x^2 - 9x + \frac{81}{4}$

$x^2 - 9x + \frac{81}{4} = \left(x - \frac{9}{2}\right)^2$

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Complete the square for the expression. Then factor the trinomial.

1. $x^2 + 10x$ **2.** $x^2 - 4x$ **3.** $x^2 + 7x$

Solving Quadratic Equations by Completing the Square

The method of completing the square can be used to solve any quadratic equation. To solve a quadratic equation by completing the square, you must write the equation in the form $x^2 + bx = d$.

COMMON ERROR

When completing the square to solve an equation, be sure to add $\left(\frac{b}{2}\right)^2$ to each side of the equation.

EXAMPLE 2 Solving a Quadratic Equation: $x^2 + bx = d$

Solve $x^2 - 16x = -15$ by completing the square.

SOLUTION

$x^2 - 16x = -15$	Write the equation.
$x^2 - 16x + (-8)^2 = -15 + (-8)^2$	Complete the square by adding $\left(\frac{-16}{2}\right)^2$, or $(-8)^2$, to each side.
$(x - 8)^2 = 49$	Write the left side as the square of a binomial.
$x - 8 = \pm 7$	Take the square root of each side.
$x = 8 \pm 7$	Add 8 to each side.

▶ The solutions are $x = 8 + 7 = 15$ and $x = 8 - 7 = 1$.

Check

$x^2 - 16x = -15$	Original equation	$x^2 - 16x = -15$
$15^2 - 16(15) \stackrel{?}{=} -15$	Substitute.	$1^2 - 16(1) \stackrel{?}{=} -15$
$-15 = -15$ ✓	Simplify.	$-15 = -15$ ✓

COMMON ERROR

Before you complete the square, be sure that the coefficient of the x^2-term is 1.

EXAMPLE 3 Solving a Quadratic Equation: $ax^2 + bx + c = 0$

Solve $2x^2 + 20x - 8 = 0$ by completing the square.

SOLUTION

$2x^2 + 20x - 8 = 0$	Write the equation.
$2x^2 + 20x = 8$	Add 8 to each side.
$x^2 + 10x = 4$	Divide each side by 2.
$x^2 + 10x + 5^2 = 4 + 5^2$	Complete the square by adding $\left(\frac{10}{2}\right)^2$, or 5^2, to each side.
$(x + 5)^2 = 29$	Write the left side as the square of a binomial.
$x + 5 = \pm\sqrt{29}$	Take the square root of each side.
$x = -5 \pm \sqrt{29}$	Subtract 5 from each side.

▶ The solutions are $x = -5 + \sqrt{29} \approx 0.39$ and $x = -5 - \sqrt{29} \approx -10.39$.

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Solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary.

4. $x^2 - 2x = 3$ **5.** $m^2 + 12m = -8$ **6.** $3g^2 - 24g + 27 = 0$

Finding and Using Maximum and Minimum Values

One way to find the maximum or minimum value of a quadratic function is to write the function in vertex form by completing the square. Recall that the vertex form of a quadratic function is $y = a(x - h)^2 + k$, where $a \neq 0$. The vertex of the graph is (h, k).

EXAMPLE 4 Finding a Minimum Value

Find the minimum value of $y = x^2 + 4x - 1$.

SOLUTION

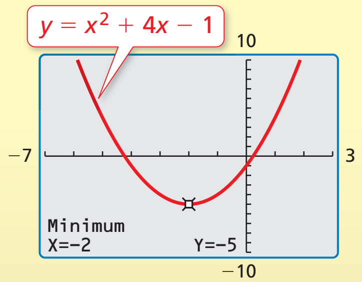
Write the function in vertex form.

$y = x^2 + 4x - 1$	Write the function.
$y + 1 = x^2 + 4x$	Add 1 to each side.
$y + 1 + 4 = x^2 + 4x + 4$	Complete the square for $x^2 + 4x$.
$y + 5 = x^2 + 4x + 4$	Simplify the left side.
$y + 5 = (x + 2)^2$	Write the right side as the square of a binomial.
$y = (x + 2)^2 - 5$	Write in vertex form.

The vertex is $(-2, -5)$. Because a is positive $(a = 1)$, the parabola opens up and the y-coordinate of the vertex is the minimum value.

▶ So, the function has a minimum value of -5.

Check



EXAMPLE 5 Finding a Maximum Value

Find the maximum value of $y = -x^2 + 2x + 7$.

SOLUTION

Write the function in vertex form.

$y = -x^2 + 2x + 7$	Write the function.
$y - 7 = -x^2 + 2x$	Subtract 7 from each side.
$y - 7 = -(x^2 - 2x)$	Factor out -1.
$y - 7 - 1 = -(x^2 - 2x + 1)$	Complete the square for $x^2 - 2x$.
$y - 8 = -(x^2 - 2x + 1)$	Simplify the left side.
$y - 8 = -(x - 1)^2$	Write $x^2 - 2x + 1$ as the square of a binomial.
$y = -(x - 1)^2 + 8$	Write in vertex form.

STUDY TIP

Adding 1 inside the parentheses results in subtracting 1 from the right side of the equation.

The vertex is $(1, 8)$. Because a is negative $(a = -1)$, the parabola opens down and the y-coordinate of the vertex is the maximum value.

▶ So, the function has a maximum value of 8.

Monitoring Progress

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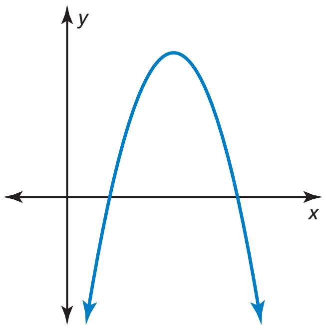
Determine whether the quadratic function has a maximum or minimum value. Then find the value.

7. $y = -x^2 - 4x + 4$ **8.** $y = x^2 + 12x + 40$ **9.** $y = x^2 - 2x - 2$

EXAMPLE 6 Interpreting Forms of Quadratic Functions

Which of the functions could be represented by the graph? Explain.



$f(x) = -\frac{1}{2}(x + 4)^2 + 8$

$g(x) = -(x - 5)^2 + 9$

$m(x) = (x - 3)(x - 12)$

$p(x) = -(x - 2)(x - 8)$

SOLUTION

You do not know the scale of either axis. To eliminate functions, consider the characteristics of the graph and information provided by the form of each function. The graph appears to be a parabola that opens down, which means the function has a maximum value. The vertex of the graph is in the first quadrant. Both x-intercepts are positive.

- The graph of f opens down because $a < 0$, which means f has a maximum value. However, the vertex $(-4, 8)$ of the graph of f is in the second quadrant. So, the graph does not represent f.
- The graph of g opens down because $a < 0$, which means g has a maximum value. The vertex $(5, 9)$ of the graph of g is in the first quadrant. By solving $0 = -(x - 5)^2 + 9$, you see that the x-intercepts of the graph of g are 2 and 8. So, the graph could represent g.
- The graph of m has two positive x-intercepts. However, its graph opens up because $a > 0$, which means m has a minimum value. So, the graph does not represent m.
- The graph of p has two positive x-intercepts, and its graph opens down because $a < 0$. This means that p has a maximum value and the vertex must be in the first quadrant. So, the graph could represent p.

▶ The graph could represent function g or function p.

EXAMPLE 7 Real-Life Application

The function $y = -16x^2 + 96x$ represents the height y (in feet) of a model rocket x seconds after it is launched. (a) Find the maximum height of the rocket. (b) Find and interpret the axis of symmetry.

SOLUTION

STUDY TIP

Adding 9 inside the parentheses results in subtracting 144 from the right side of the equation.

a. To find the maximum height, identify the maximum value of the function.

$y = -16x^2 + 96x$	Write the function.
$y = -16(x^2 - 6x)$	Factor out -16.
$y - 144 = -16(x^2 - 6x + 9)$	Complete the square for $x^2 - 6x$.
$y = -16(x - 3)^2 + 144$	Write in vertex form.

▶ Because the maximum value is 144, the model rocket reaches a maximum height of 144 feet.

b. The vertex is $(3, 144)$. So, the axis of symmetry is $x = 3$. On the left side of $x = 3$, the height increases as time increases. On the right side of $x = 3$, the height decreases as time increases.

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Determine whether the function could be represented by the graph in Example 6. Explain.

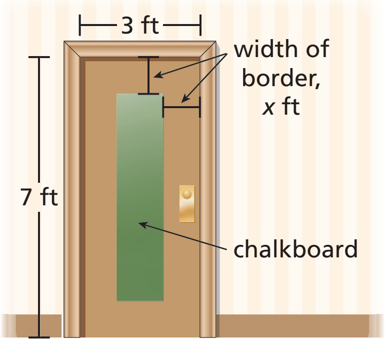
10. $h(x) = (x - 8)^2 + 10$

11. $n(x) = -2(x - 5)(x - 20)$

12. WHAT IF? Repeat Example 7 when the function is $y = -16x^2 + 128x$.

Solving Real-Life Problems

EXAMPLE 8 **Modeling with Mathematics**



You decide to use chalkboard paint to create a chalkboard on a door. You want the chalkboard to cover 6 square feet and to have a uniform border, as shown. Find the width of the border to the nearest inch.

SOLUTION

1. **Understand the Problem** You know the dimensions (in feet) of the door from the diagram. You also know the area (in square feet) of the chalkboard and that it will have a uniform border. You are asked to find the width of the border to the nearest inch.

2. **Make a Plan** Use a verbal model to write an equation that represents the area of the chalkboard. Then solve the equation.

3. **Solve the Problem**

Let x be the width (in feet) of the border, as shown in the diagram.

Area of chalkboard (square feet)	=	Length of chalkboard (feet)	•	Width of chalkboard (feet)
6	=	$(7 - 2x)$	•	$(3 - 2x)$

$6 = (7 - 2x)(3 - 2x)$ Write the equation.

$6 = 21 - 20x + 4x^2$ Multiply the binomials.

$-15 = 4x^2 - 20x$ Subtract 21 from each side.

$-\frac{15}{4} = x^2 - 5x$ Divide each side by 4.

$-\frac{15}{4} + \frac{25}{4} = x^2 - 5x + \frac{25}{4}$ Complete the square for $x^2 - 5x$.

$\frac{5}{2} = x^2 - 5x + \frac{25}{4}$ Simplify the left side.

$\frac{5}{2} = \left(x - \frac{5}{2}\right)^2$ Write the right side as the square of a binomial.

$\pm\sqrt{\frac{5}{2}} = x - \frac{5}{2}$ Take the square root of each side.

$\frac{5}{2} \pm \sqrt{\frac{5}{2}} = x$ Add $\frac{5}{2}$ to each side.

The solutions of the equation are $x = \frac{5}{2} + \sqrt{\frac{5}{2}} \approx 4.08$ and $x = \frac{5}{2} - \sqrt{\frac{5}{2}} \approx 0.92$.

It is not possible for the width of the border to be 4.08 feet because the width of the door is 3 feet. So, the width of the border is about 0.92 foot.

$0.92\ \cancel{\text{ft}} \cdot \frac{12\text{ in.}}{1\ \cancel{\text{ft}}} = 11.04\text{ in.}$ Convert 0.92 foot to inches.

▶ The width of the border should be about 11 inches.

4. **Look Back** When the width of the border is slightly less than 1 foot, the length of the chalkboard is slightly more than 5 feet and the width of the chalkboard is slightly more than 1 foot. Multiplying these dimensions gives an area close to 6 square feet. So, an 11-inch border is reasonable.

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13. **WHAT IF?** You want the chalkboard to cover 4 square feet. Find the width of the border to the nearest inch.

4.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The process of adding a constant c to the expression $x^2 + bx$ so that $x^2 + bx + c$ is a perfect square trinomial is called ________________.

2. **VOCABULARY** Explain how to complete the square for an expression of the form $x^2 + bx$.

3. **WRITING** Is it more convenient to complete the square for $x^2 + bx$ when b is odd or when b is even? Explain.

4. **WRITING** Describe how you can use the process of completing the square to find the maximum or minimum value of a quadratic function.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, find the value of c that completes the square.

5. $x^2 - 8x + c$
6. $x^2 - 2x + c$
7. $x^2 + 4x + c$
8. $x^2 + 12x + c$
9. $x^2 - 15x + c$
10. $x^2 + 9x + c$

In Exercises 11–16, complete the square for the expression. Then factor the trinomial. *(See Example 1.)*

11. $x^2 - 10x$
12. $x^2 - 40x$
13. $x^2 + 16x$
14. $x^2 + 22x$
15. $x^2 + 5x$
16. $x^2 - 3x$

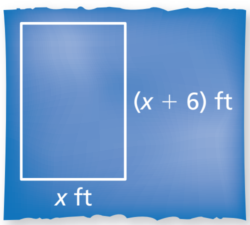
In Exercises 17–22, solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary. *(See Example 2.)*

17. $x^2 + 14x = 15$
18. $x^2 - 6x = 16$
19. $x^2 - 4x = -2$
20. $x^2 + 2x = 5$
21. $x^2 - 5x = 8$
22. $x^2 + 11x = -10$

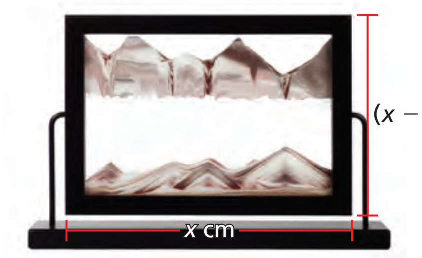
23. **MODELING WITH MATHEMATICS** The area of the patio is 216 square feet.

    a. Write an equation that represents the area of the patio.

    b. Find the dimensions of the patio by completing the square.



24. **MODELING WITH MATHEMATICS** Some sand art contains sand and water sealed in a glass case, similar to the one shown. When the art is turned upside down, the sand and water fall to create a new picture. The glass case has a depth of 1 centimeter and a volume of 768 cubic centimeters.



    a. Write an equation that represents the volume of the glass case.

    b. Find the dimensions of the glass case by completing the square.

In Exercises 25–32, solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary. *(See Example 3.)*

25. $x^2 - 8x + 15 = 0$
26. $x^2 + 4x - 21 = 0$
27. $2x^2 + 20x + 44 = 0$
28. $3x^2 - 18x + 12 = 0$
29. $-3x^2 - 24x + 17 = -40$
30. $-5x^2 - 20x + 35 = 30$
31. $2x^2 - 14x + 10 = 26$
32. $4x^2 + 12x - 15 = 5$

33. ERROR ANALYSIS Describe and correct the error in solving $x^2 + 8x = 10$ by completing the square.

$$\begin{aligned} x^2 + 8x &= 10 \\ x^2 + 8x + 16 &= 10 \\ (x + 4)^2 &= 10 \\ x + 4 &= \pm\sqrt{10} \\ x &= -4 \pm \sqrt{10} \end{aligned}$$

34. ERROR ANALYSIS Describe and correct the error in the first two steps of solving $2x^2 - 2x - 4 = 0$ by completing the square.

$$\begin{aligned} 2x^2 - 2x - 4 &= 0 \\ 2x^2 - 2x &= 4 \\ 2x^2 - 2x + 1 &= 4 + 1 \end{aligned}$$

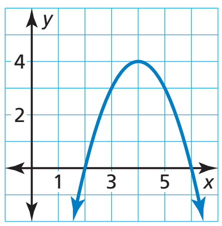
35. NUMBER SENSE Find all values of b for which $x^2 + bx + 25$ is a perfect square trinomial. Explain how you found your answer.

36. REASONING You are completing the square to solve $3x^2 + 6x = 12$. What is the first step?

In Exercises 37–40, write the function in vertex form by completing the square. Then match the function with its graph.

37. $y = x^2 + 6x + 3$ **38.** $y = -x^2 + 8x - 12$

39. $y = -x^2 - 4x - 2$ **40.** $y = x^2 - 2x + 4$

A.


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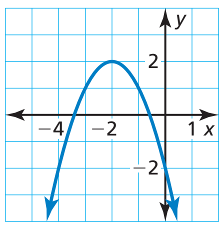

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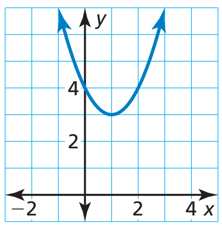
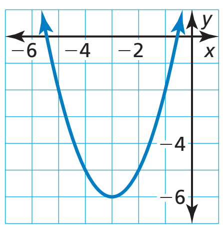


D.


In Exercises 41–46, determine whether the quadratic function has a maximum or minimum value. Then find the value. *(See Examples 4 and 5.)*

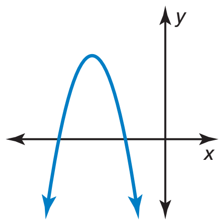
41. $y = x^2 - 4x - 2$ **42.** $y = x^2 + 6x + 10$

43. $y = -x^2 - 10x - 30$ **44.** $y = -x^2 + 14x - 34$

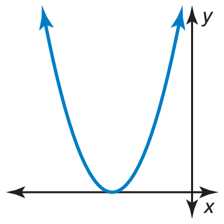
45. $f(x) = -3x^2 - 6x - 9$ **46.** $f(x) = 4x^2 - 28x + 32$

In Exercises 47–50, determine whether the graph could represent the function. Explain.

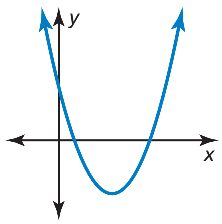
47. $y = -(x + 8)(x + 3)$



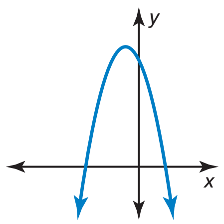
48. $y = (x - 5)^2$



49. $y = \frac{1}{4}(x + 2)^2 - 4$

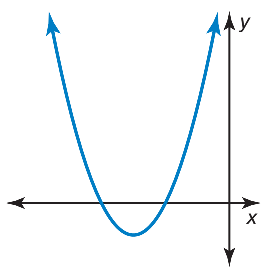


50. $y = -2(x - 1)(x + 2)$



In Exercises 51 and 52, determine which of the functions could be represented by the graph. Explain. *(See Example 6.)*

51.



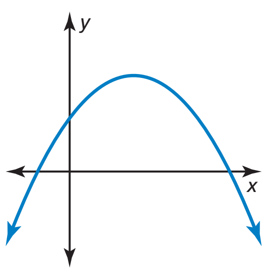
$h(x) = (x + 2)^2 + 3$

$f(x) = 2(x + 3)^2 - 2$

$g(x) = -\frac{1}{2}(x - 8)(x - 4)$

$m(x) = (x + 2)(x + 4)$

52.



$r(x) = -\frac{1}{3}(x - 5)(x + 1)$

$p(x) = -2(x - 2)(x - 6)$

$q(x) = (x + 1)^2 + 4$

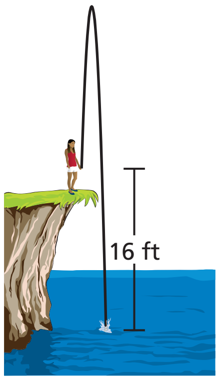
$n(x) = -(x - 2)^2 + 9$

53. MODELING WITH MATHEMATICS The function $h = -16t^2 + 48t$ represents the height h (in feet) of a kickball t seconds after it is kicked from the ground. *(See Example 7.)*

a. Find the maximum height of the kickball.

b. Find and interpret the axis of symmetry.

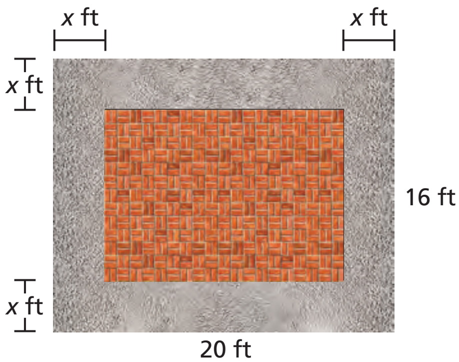
54. MODELING WITH MATHEMATICS You throw a stone from a height of 16 feet with an initial vertical velocity of 32 feet per second. The function $h = -16t^2 + 32t + 16$ represents the height h (in feet) of the stone after t seconds.



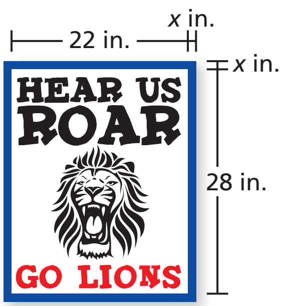
a. Find the maximum height of the stone.

b. Find and interpret the axis of symmetry.

55. MODELING WITH MATHEMATICS You are building a rectangular brick patio surrounded by a crushed stone border with a uniform width, as shown. You purchase patio bricks to cover 140 square feet. Find the width of the border. *(See Example 8.)*

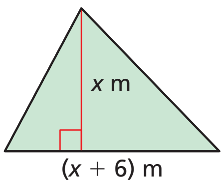


56. MODELING WITH MATHEMATICS You are making a poster that will have a uniform border, as shown. The total area of the poster is 722 square inches. Find the width of the border to the nearest inch.

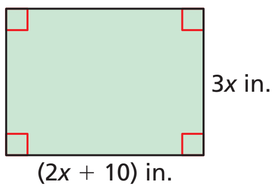


MATHEMATICAL CONNECTIONS **In Exercises 57 and 58, find the value of x. Round your answer to the nearest hundredth, if necessary.**

57. $A = 108 \text{ m}^2$



58. $A = 288 \text{ in.}^2$



In Exercises 59–62, solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary.

59. $0.5x^2 + x - 2 = 0$

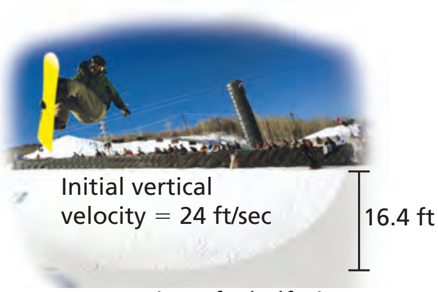
60. $0.75x^2 + 1.5x = 4$

61. $\frac{8}{3}x - \frac{2}{3}x^2 = -\frac{5}{6}$

62. $\frac{1}{4}x^2 + \frac{1}{2}x - \frac{5}{4} = 0$

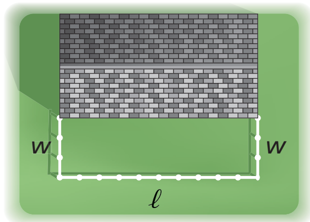
63. PROBLEM SOLVING The distance d (in feet) that it takes a car to come to a complete stop can be modeled by $d = 0.05s^2 + 2.2s$, where s is the speed of the car (in miles per hour). A car has 168 feet to come to a complete stop. Find the maximum speed at which the car can travel.

64. PROBLEM SOLVING During a "big air" competition, snowboarders launch themselves from a half-pipe, perform tricks in the air, and land back in the half-pipe. The height h (in feet) of a snowboarder above the bottom of the half-pipe can be modeled by $h = -16t^2 + 24t + 16.4$, where t is the time (in seconds) after the snowboarder launches into the air. The snowboarder lands 3.2 feet lower than the height of the launch. How long is the snowboarder in the air? Round your answer to the nearest tenth of a second.



Cross section of a half-pipe

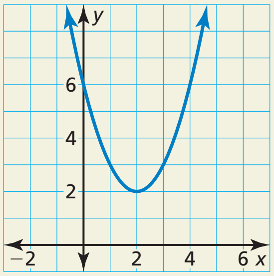
65. PROBLEM SOLVING You have 80 feet of fencing to make a rectangular horse pasture that covers 750 square feet. A barn will be used as one side of the pasture, as shown.



a. Write equations for the amount of fencing to be used and the area enclosed by the fencing.

b. Use substitution to solve the system of equations from part (a). What are the possible dimensions of the pasture?

66. **HOW DO YOU SEE IT?** The graph represents the quadratic function $y = x^2 - 4x + 6$.



a. Use the graph to estimate the x-values for which $y = 3$.

b. Explain how you can use the method of completing the square to check your estimates in part (a).

67. **COMPARING METHODS** Consider the quadratic equation $x^2 + 12x + 2 = 12$.

a. Solve the equation by graphing.

b. Solve the equation by completing the square.

c. Compare the two methods. Which do you prefer? Explain.

68. **THOUGHT PROVOKING** Sketch the graph of the equation $x^2 - 2xy + y^2 - x - y = 0$. Identify the graph.

69. **REASONING** The product of two consecutive even integers that are positive is 48. Write and solve an equation to find the integers.

70. **REASONING** The product of two consecutive odd integers that are negative is 195. Write and solve an equation to find the integers.

71. **MAKING AN ARGUMENT** You purchase stock for \$16 per share. You sell the stock 30 days later for \$23.50 per share. The price y (in dollars) of a share during the 30-day period can be modeled by $y = -0.025x^2 + x + 16$, where x is the number of days after the stock is purchased. Your friend says you could have sold the stock earlier for \$23.50 per share. Is your friend correct? Explain.

72. **REASONING** You are solving the equation $x^2 + 9x = 18$. What are the advantages of solving the equation by completing the square instead of using other methods you have learned?

73. **PROBLEM SOLVING** You are knitting a rectangular scarf. The pattern results in a scarf that is 60 inches long and 4 inches wide. However, you have enough yarn to knit 396 square inches. You decide to increase the dimensions of the scarf so that you will use all your yarn. The increase in the length is three times the increase in the width. What are the dimensions of your scarf?

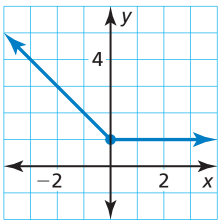
74. **WRITING** How many solutions does $x^2 + bx = c$ have when $c < -\left(\frac{b}{2}\right)^2$? Explain.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write a piecewise function for the graph. *(Section 1.2)*

75.



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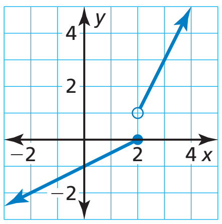

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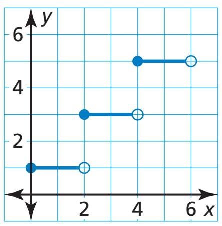

Simplify the expression $\sqrt{b^2 - 4ac}$ for the given values. *(Section 4.1)*

78. $a = 3, b = -6, c = 2$

79. $a = -2, b = 4, c = 7$

80. $a = 1, b = 6, c = 4$

4.5 Solving Quadratic Equations Using the Quadratic Formula

Essential Question How can you derive a formula that can be used to write the solutions of any quadratic equation in standard form?

EXPLORATION 1 Deriving the Quadratic Formula

Work with a partner. The following steps show a method of solving $ax^2 + bx + c = 0$. Explain what was done in each step.

$ax^2 + bx + c = 0$	1. Write the equation.
$4a^2x^2 + 4abx + 4ac = 0$	2. What was done?
$4a^2x^2 + 4abx + 4ac + b^2 = b^2$	3. What was done?
$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$	4. What was done?
$(2ax + b)^2 = b^2 - 4ac$	5. What was done?
$2ax + b = \pm\sqrt{b^2 - 4ac}$	6. What was done?
$2ax = -b \pm \sqrt{b^2 - 4ac}$	7. What was done?
Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	8. What was done?

EXPLORATION 2 Deriving the Quadratic Formula by Completing the Square

Work with a partner.

a. Solve $ax^2 + bx + c = 0$ by completing the square. (*Hint:* Subtract c from each side, divide each side by a, and then proceed by completing the square.)

b. Compare this method with the method in Exploration 1. Explain why you think $4a$ and b^2 were chosen in Steps 2 and 3 of Exploration 1.

Communicate Your Answer

3. How can you derive a formula that can be used to write the solutions of any quadratic equation in standard form?

4. Use the Quadratic Formula to solve each quadratic equation.

a. $x^2 + 2x - 3 = 0$ **b.** $x^2 - 4x + 4 = 0$ **c.** $x^2 + 4x + 5 = 0$

5. Use the Internet to research *imaginary numbers*. How are they related to quadratic equations?

USING TOOLS STRATEGICALLY

To be proficient in math, you need to identify relevant external mathematical resources.

4.5 Lesson

Core Vocabulary

Quadratic Formula, *p. 226*
discriminant, *p. 228*

What You Will Learn

- Solve quadratic equations using the Quadratic Formula.
- Interpret the discriminant.
- Choose efficient methods for solving quadratic equations.

Using the Quadratic Formula

By completing the square for the quadratic equation $ax^2 + bx + c = 0$, you can develop a formula that gives the solutions of any quadratic equation in standard form. This formula is called the **Quadratic Formula**.

Core Concept

Quadratic Formula

The real solutions of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

where $a \neq 0$ and $b^2 - 4ac \geq 0$.

EXAMPLE 1 Using the Quadratic Formula

Solve $2x^2 - 5x + 3 = 0$ using the Quadratic Formula.

SOLUTION

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(3)}}{2(2)} \qquad \text{Substitute 2 for } a, -5 \text{ for } b, \text{ and 3 for } c.$$

$$= \frac{5 \pm \sqrt{1}}{4} \qquad \text{Simplify.}$$

$$= \frac{5 \pm 1}{4} \qquad \text{Evaluate the square root.}$$

So, the solutions are $x = \frac{5+1}{4} = \frac{3}{2}$ and $x = \frac{5-1}{4} = 1$.

STUDY TIP

You can use the roots of a quadratic equation to factor the related expression. In Example 1, you can use 1 and $\frac{3}{2}$ to factor $2x^2 - 5x + 3$ as $(x - 1)(2x - 3)$.

Check

$2x^2 - 5x + 3 = 0$	Original equation	$2x^2 - 5x + 3 = 0$
$2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3 \stackrel{?}{=} 0$	Substitute.	$2(1)^2 - 5(1) + 3 \stackrel{?}{=} 0$
$\frac{9}{2} - \frac{15}{2} + 3 \stackrel{?}{=} 0$	Simplify.	$2 - 5 + 3 \stackrel{?}{=} 0$
$0 = 0$ ✓	Simplify.	$0 = 0$ ✓

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Solve the equation using the Quadratic Formula. Round your solutions to the nearest tenth, if necessary.

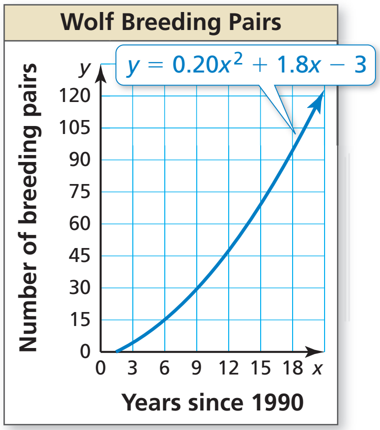
1. $x^2 - 6x + 5 = 0$

2. $\frac{1}{2}x^2 + x - 10 = 0$

3. $-3x^2 + 2x + 7 = 0$

4. $4x^2 - 4x = -1$

EXAMPLE 2 Modeling With Mathematics



The number y of Northern Rocky Mountain wolf breeding pairs x years since 1990 can be modeled by the function $y = 0.20x^2 + 1.8x - 3$. When were there about 35 breeding pairs?

SOLUTION

1. **Understand the Problem** You are given a quadratic function that represents the number of wolf breeding pairs for years after 1990. You need to use the model to determine when there were 35 wolf breeding pairs.

2. **Make a Plan** To determine when there were 35 wolf breeding pairs, find the x-values for which $y = 35$. So, solve the equation $35 = 0.20x^2 + 1.8x - 3$.

3. **Solve the Problem**

$$35 = 0.20x^2 + 1.8x - 3 \quad \text{Write the equation.}$$

$$0 = 0.20x^2 + 1.8x - 38 \quad \text{Write in standard form.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Quadratic Formula}$$

$$= \frac{-1.8 \pm \sqrt{1.8^2 - 4(0.2)(-38)}}{2(0.2)} \quad \text{Substitute 0.2 for } a\text{, 1.8 for } b\text{, and } -38 \text{ for } c.$$

$$= \frac{-1.8 \pm \sqrt{33.64}}{0.4} \quad \text{Simplify.}$$

$$= \frac{-1.8 \pm 5.8}{0.4} \quad \text{Simplify.}$$

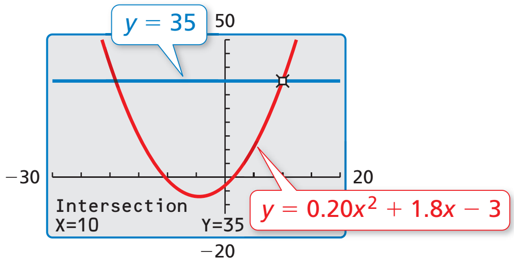
The solutions are $x = \frac{-1.8 + 5.8}{0.4} = 10$ and $x = \frac{-1.8 - 5.8}{0.4} = -19$.

INTERPRETING MATHEMATICAL RESULTS

You can ignore the solution $x = -19$ because -19 represents the year 1971, which is not in the given time period.

▶ Because x represents the number of years since 1990, x is greater than or equal to zero. So, there were about 35 breeding pairs 10 years after 1990, in 2000.

4. **Look Back** Use a graphing calculator to graph the equations $y = 0.20x^2 + 1.8x - 3$ and $y = 35$. Then use the *intersect* feature to find the point of intersection. The graphs intersect at (10, 35).



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5. **WHAT IF?** When were there about 60 wolf breeding pairs?

6. The number y of bald eagle nesting pairs in a state x years since 2000 can be modeled by the function $y = 0.34x^2 + 13.1x + 51$.

   a. When were there about 160 bald eagle nesting pairs?

   b. How many bald eagle nesting pairs were there in 2000?

Interpreting the Discriminant

The expression $b^2 - 4ac$ in the Quadratic Formula is called the **discriminant**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

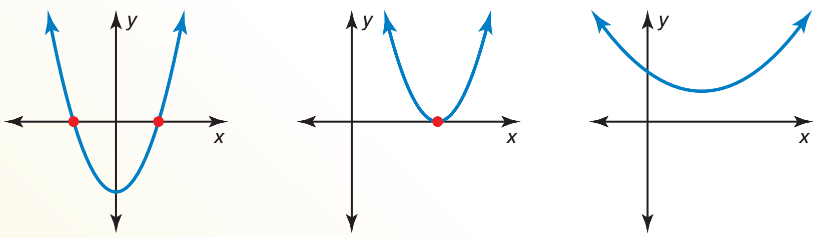
discriminant

Because the discriminant is under the radical symbol, you can use the value of the discriminant to determine the number of real solutions of a quadratic equation and the number of x-intercepts of the graph of the related function.

Core Concept

Interpreting the Discriminant

$b^2 - 4ac > 0$ | $b^2 - 4ac = 0$ | $b^2 - 4ac < 0$



$b^2 - 4ac > 0$:
- two real solutions
- two x-intercepts

$b^2 - 4ac = 0$:
- one real solution
- one x-intercept

$b^2 - 4ac < 0$:
- no real solutions
- no x-intercepts

EXAMPLE 3 Determining the Number of Real Solutions

a. Determine the number of real solutions of $x^2 + 8x - 3 = 0$.

$$\begin{aligned} b^2 - 4ac &= 8^2 - 4(1)(-3) && \text{Substitute 1 for } a\text{, 8 for } b\text{, and } -3 \text{ for } c. \\ &= 64 + 12 && \text{Simplify.} \\ &= 76 && \text{Add.} \end{aligned}$$

▶ The discriminant is greater than 0. So, the equation has two real solutions.

b. Determine the number of real solutions of $9x^2 + 1 = 6x$.

Write the equation in standard form: $9x^2 - 6x + 1 = 0$.

$$\begin{aligned} b^2 - 4ac &= (-6)^2 - 4(9)(1) && \text{Substitute 9 for } a\text{, } -6 \text{ for } b\text{, and 1 for } c. \\ &= 36 - 36 && \text{Simplify.} \\ &= 0 && \text{Subtract.} \end{aligned}$$

▶ The discriminant is 0. So, the equation has one real solution.

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Determine the number of real solutions of the equation.

7. $-x^2 + 4x - 4 = 0$

8. $6x^2 + 2x = -1$

9. $\frac{1}{2}x^2 = 7x - 1$

EXAMPLE 4 Finding the Number of x-Intercepts of a Parabola

Find the number of x-intercepts of the graph of $y = 2x^2 + 3x + 9$.

SOLUTION

Determine the number of real solutions of $0 = 2x^2 + 3x + 9$.

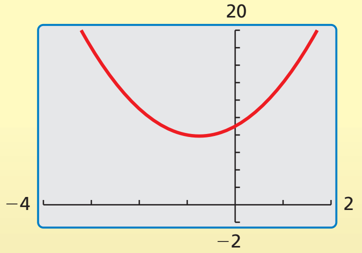
$$b^2 - 4ac = 3^2 - 4(2)(9) \qquad \text{Substitute 2 for } a\text{, 3 for } b\text{, and 9 for } c.$$
$$= 9 - 72 \qquad \text{Simplify.}$$
$$= -63 \qquad \text{Subtract.}$$

Because the discriminant is less than 0, the equation has no real solutions.

▶ So, the graph of $y = 2x^2 + 3x + 9$ has no x-intercepts.

Check

Use a graphing calculator to check your answer. Notice that the graph of $y = 2x^2 + 3x + 9$ has no x-intercepts.



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Find the number of x-intercepts of the graph of the function.

10. $y = -x^2 + x - 6$ **11.** $y = x^2 - x$ **12.** $f(x) = x^2 + 12x + 36$

Choosing an Efficient Method

The table shows five methods for solving quadratic equations. For a given equation, it may be more efficient to use one method instead of another. Some advantages and disadvantages of each method are shown.

Core Concept

Methods for Solving Quadratic Equations

Method	Advantages	Disadvantages
Factoring *(Lessons 2.5–2.8)*	• Straightforward when the equation can be factored easily	• Some equations are not factorable.
Graphing *(Lesson 4.2)*	• Can easily see the number of solutions • Use when approximate solutions are sufficient. • Can use a graphing calculator	• May not give exact solutions
Using Square Roots *(Lesson 4.3)*	• Use to solve equations of the form $x^2 = d$.	• Can only be used for certain equations
Completing the Square *(Lesson 4.4)*	• Best used when $a = 1$ and b is even	• May involve difficult calculations
Quadratic Formula *(Lesson 4.5)*	• Can be used for any quadratic equation • Gives exact solutions	• Takes time to do calculations

EXAMPLE 5 Choosing a Method

Solve the equation using any method. Explain your choice of method.

a. $x^2 - 10x = 1$ **b.** $2x^2 - 13x - 24 = 0$ **c.** $x^2 + 8x + 12 = 0$

SOLUTION

a. The coefficient of the x^2-term is 1, and the coefficient of the x-term is an even number. So, solve by completing the square.

$$x^2 - 10x = 1 \qquad \text{Write the equation.}$$

$$x^2 - 10x + 25 = 1 + 25 \qquad \text{Complete the square for } x^2 - 10x.$$

$$(x - 5)^2 = 26 \qquad \text{Write the left side as the square of a binomial.}$$

$$x - 5 = \pm\sqrt{26} \qquad \text{Take the square root of each side.}$$

$$x = 5 \pm \sqrt{26} \qquad \text{Add 5 to each side.}$$

▶ So, the solutions are $x = 5 + \sqrt{26} \approx 10.1$ and $x = 5 - \sqrt{26} \approx -0.1$.

b. The equation is not easily factorable, and the numbers are somewhat large. So, solve using the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$= \frac{-(-13) \pm \sqrt{(-13)^2 - 4(2)(-24)}}{2(2)} \qquad \text{Substitute 2 for } a, -13 \text{ for } b, \text{ and } -24 \text{ for } c.$$

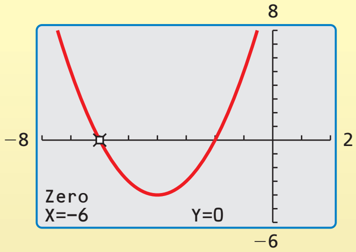
$$= \frac{13 \pm \sqrt{361}}{4} \qquad \text{Simplify.}$$

$$= \frac{13 \pm 19}{4} \qquad \text{Evaluate the square root.}$$

▶ So, the solutions are $x = \frac{13 + 19}{4} = 8$ and $x = \frac{13 - 19}{4} = -\frac{3}{2}$.

Check

Graph the related function $f(x) = x^2 + 8x + 12$ and find the zeros. The zeros are -6 and -2.



c. The equation is easily factorable. So, solve by factoring.

$$x^2 + 8x + 12 = 0 \qquad \text{Write the equation.}$$

$$(x + 2)(x + 6) = 0 \qquad \text{Factor the polynomial.}$$

$$x + 2 = 0 \quad \text{or} \quad x + 6 = 0 \qquad \text{Zero-Product Property}$$

$$x = -2 \quad \text{or} \quad x = -6 \qquad \text{Solve for } x.$$

▶ The solutions are $x = -2$ and $x = -6$.

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Solve the equation using any method. Explain your choice of method.

13. $x^2 + 11x - 12 = 0$

14. $9x^2 - 5 = 4$

15. $5x^2 - x - 1 = 0$

16. $x^2 = 2x - 5$

4.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What formula can you use to solve any quadratic equation? Write the formula.

2. **VOCABULARY** In the Quadratic Formula, what is the discriminant? What does the value of the discriminant determine?

Monitoring Progress and Modeling with Mathematics

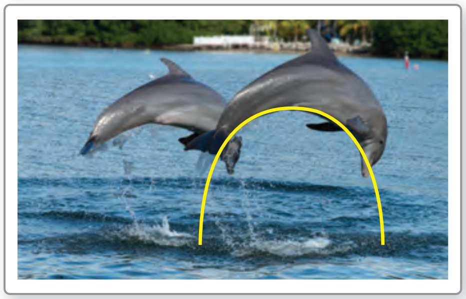
In Exercises 3–8, write the equation in standard form. Then identify the values of *a*, *b*, and *c* that you would use to solve the equation using the Quadratic Formula.

3. $x^2 = 7x$
4. $x^2 - 4x = -12$
5. $-2x^2 + 1 = 5x$
6. $3x + 2 = 4x^2$
7. $4 - 3x = -x^2 + 3x$
8. $-8x - 1 = 3x^2 + 2$

In Exercises 9–22, solve the equation using the Quadratic Formula. Round your solutions to the nearest tenth, if necessary. *(See Example 1.)*

9. $x^2 - 12x + 36 = 0$
10. $x^2 + 7x + 16 = 0$
11. $x^2 - 10x - 11 = 0$
12. $2x^2 - x - 1 = 0$
13. $2x^2 - 6x + 5 = 0$
14. $9x^2 - 6x + 1 = 0$
15. $6x^2 - 13x = -6$
16. $-3x^2 + 6x = 4$
17. $1 - 8x = -16x^2$
18. $x^2 - 5x + 3 = 0$
19. $x^2 + 2x = 9$
20. $5x^2 - 2 = 4x$
21. $2x^2 + 9x + 7 = 3$
22. $8x^2 + 8 = 6 - 9x$

23. **MODELING WITH MATHEMATICS** A dolphin jumps out of the water, as shown in the diagram. The function $h = -16t^2 + 26t$ models the height h (in feet) of the dolphin after t seconds. After how many seconds is the dolphin at a height of 5 feet? *(See Example 2.)*



24. **MODELING WITH MATHEMATICS** The amount of trout y (in tons) caught in a lake from 1995 to 2014 can be modeled by the equation $y = -0.08x^2 + 1.6x + 10$, where x is the number of years since 1995.

    a. When were about 15 tons of trout caught in the lake?

    b. Do you think this model can be used to determine the amounts of trout caught in future years? Explain your reasoning.

In Exercises 25–30, determine the number of real solutions of the equation. *(See Example 3.)*

25. $x^2 - 6x + 10 = 0$
26. $x^2 - 5x - 3 = 0$
27. $2x^2 - 12x = -18$
28. $4x^2 = 4x - 1$
29. $-\frac{1}{4}x^2 + 4x = -2$
30. $-5x^2 + 8x = 9$

In Exercises 31–36, find the number of *x*-intercepts of the graph of the function. *(See Example 4.)*

31. $y = x^2 + 5x - 1$
32. $y = 4x^2 + 4x + 1$
33. $y = -6x^2 + 3x - 4$
34. $y = -x^2 + 5x + 13$
35. $f(x) = 4x^2 + 3x - 6$
36. $f(x) = 2x^2 + 8x + 8$

In Exercises 37–44, solve the equation using any method. Explain your choice of method. *(See Example 5.)*

37. $-10x^2 + 13x = 4$
38. $x^2 - 3x - 40 = 0$
39. $x^2 + 6x = 5$
40. $-5x^2 = -25$
41. $x^2 + x - 12 = 0$
42. $x^2 - 4x + 1 = 0$
43. $4x^2 - x = 17$
44. $x^2 + 6x + 9 = 16$

45. ERROR ANALYSIS Describe and correct the error in solving the equation $3x^2 - 7x - 6 = 0$ using the Quadratic Formula.

✗

$$x = \frac{-7 \pm \sqrt{(-7)^2 - 4(3)(-6)}}{2(3)}$$

$$= \frac{-7 \pm \sqrt{121}}{6}$$

$$x = \frac{2}{3} \text{ and } x = -3$$

46. ERROR ANALYSIS Describe and correct the error in solving the equation $-2x^2 + 9x = 4$ using the Quadratic Formula.

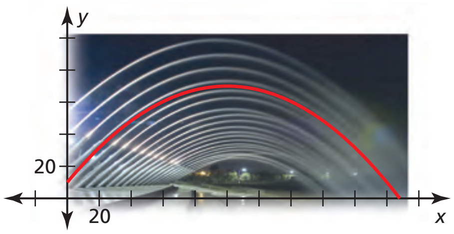
✗

$$x = \frac{-9 \pm \sqrt{9^2 - 4(-2)(4)}}{2(-2)}$$

$$= \frac{-9 \pm \sqrt{113}}{-4}$$

$$x \approx -0.41 \text{ and } x \approx 4.91$$

47. MODELING WITH MATHEMATICS A fountain shoots a water arc that can be modeled by the graph of the equation $y = -0.006x^2 + 1.2x + 10$, where x is the horizontal distance (in feet) from the river's north shore and y is the height (in feet) above the river. Does the water arc reach a height of 50 feet? If so, about how far from the north shore is the water arc 50 feet above the water?



48. MODELING WITH MATHEMATICS Between the months of April and September, the number y of hours of daylight per day in Seattle, Washington, can be modeled by $y = -0.00046x^2 + 0.076x + 13$, where x is the number of days since April 1.

a. Do any of the days between April and September in Seattle have 17 hours of daylight? If so, how many?

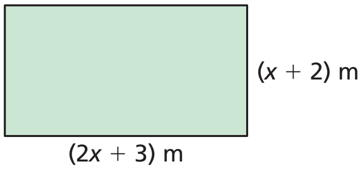
b. Do any of the days between April and September in Seattle have 14 hours of daylight? If so, how many?

49. MAKING AN ARGUMENT Your friend uses the discriminant of the equation $2x^2 - 5x - 2 = -11$ and determines that the equation has two real solutions. Is your friend correct? Explain your reasoning.

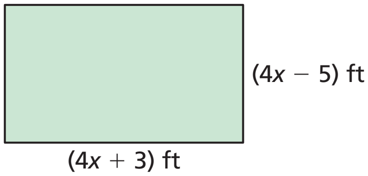
50. MODELING WITH MATHEMATICS The frame of the tent shown is defined by a rectangular base and two parabolic arches that connect the opposite corners of the base. The graph of $y = -0.18x^2 + 1.6x$ models the height y (in feet) of one of the arches x feet along the diagonal of the base. Can a child who is 4 feet tall walk under one of the arches without having to bend over? Explain.

MATHEMATICAL CONNECTIONS In Exercises 51 and 52, use the given area A of the rectangle to find the value of x. Then give the dimensions of the rectangle.

51. $A = 91 \text{ m}^2$



52. $A = 209 \text{ ft}^2$



COMPARING METHODS In Exercises 53 and 54, solve the equation by (a) graphing, (b) factoring, and (c) using the Quadratic Formula. Which method do you prefer? Explain your reasoning.

53. $x^2 + 4x + 4 = 0$

54. $3x^2 + 11x + 6 = 0$

55. REASONING How many solutions does the equation $ax^2 + bx + c = 0$ have when a and c have different signs? Explain your reasoning.

56. REASONING When the discriminant is a perfect square, are the solutions of $ax^2 + bx + c = 0$ rational or irrational? (Assume a, b, and c are integers.) Explain your reasoning.

REASONING In Exercises 57–59, give a value of c for which the equation has (a) two solutions, (b) one solution, and (c) no solutions.

57. $x^2 - 2x + c = 0$

58. $x^2 - 8x + c = 0$

59. $4x^2 + 12x + c = 0$

60. **REPEATED REASONING** You use the Quadratic Formula to solve an equation.

a. You obtain solutions that are integers. Could you have used factoring to solve the equation? Explain your reasoning.

b. You obtain solutions that are fractions. Could you have used factoring to solve the equation? Explain your reasoning.

c. Make a generalization about quadratic equations with rational solutions.

61. **MODELING WITH MATHEMATICS** The fuel economy y (in miles per gallon) of a car can be modeled by the equation $y = -0.013x^2 + 1.25x + 5.6$, where $5 \le x \le 75$ and x is the speed (in miles per hour) of the car. Find the speed(s) at which you can travel and have a fuel economy of 32 miles per gallon.

62. **MODELING WITH MATHEMATICS** The depth d (in feet) of a river can be modeled by the equation $d = -0.25t^2 + 1.7t + 3.5$, where $0 \le t \le 7$ and t is the time (in hours) after a heavy rain begins. When is the river 6 feet deep?

ANALYZING EQUATIONS **In Exercises 63–68, tell whether the vertex of the graph of the function lies above, below, or on the x-axis. Explain your reasoning without using a graph.**

63. $y = x^2 - 3x + 2$

64. $y = 3x^2 - 6x + 3$

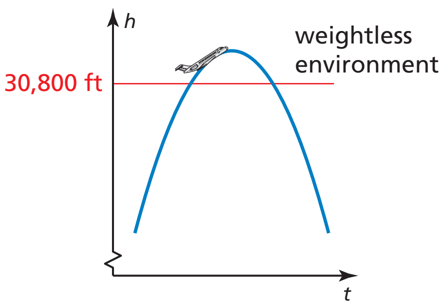
65. $y = 6x^2 - 2x + 4$

66. $y = -15x^2 + 10x - 25$

67. $f(x) = -3x^2 - 4x + 8$

68. $f(x) = 9x^2 - 24x + 16$

69. **REASONING** NASA creates a weightless environment by flying a plane in a series of parabolic paths. The height h (in feet) of a plane after t seconds in a parabolic flight path can be modeled by $h = -11t^2 + 700t + 21{,}000$. The passengers experience a weightless environment when the height of the plane is greater than or equal to 30,800 feet. For approximately how many seconds do passengers experience weightlessness on such a flight? Explain.

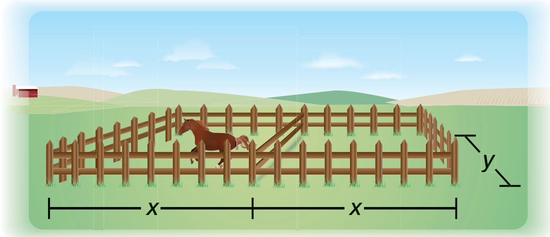


70. **WRITING EQUATIONS** Use the numbers to create a quadratic equation with the solutions $x = -1$ and $x = -\frac{1}{4}$.

$$\_\_\_x^2 + \_\_\_x + \_\_\_ = 0$$

−5	−4	−3	−2	−1
1	2	3	4	5

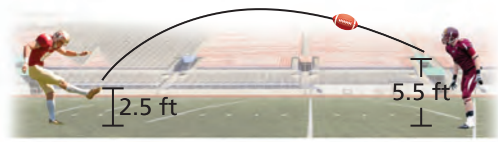
71. **PROBLEM SOLVING** A rancher constructs two rectangular horse pastures that share a side, as shown. The pastures are enclosed by 1050 feet of fencing. Each pasture has an area of 15,000 square feet.



a. Show that $y = 350 - \frac{4}{3}x$.

b. Find the possible lengths and widths of each pasture.

72. **PROBLEM SOLVING** A kicker punts a football from a height of 2.5 feet above the ground with an initial vertical velocity of 45 feet per second.



Not drawn to scale

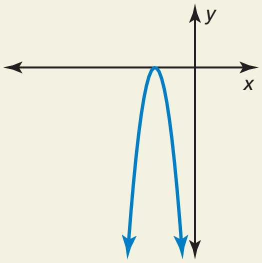
a. Write an equation that models this situation using the function $h = -16t^2 + v_0t + s_0$, where h is the height (in feet) of the football, t is the time (in seconds) after the football is punted, v_0 is the initial vertical velocity (in feet per second), and s_0 is the initial height (in feet).

b. The football is caught 5.5 feet above the ground, as shown in the diagram. Find the amount of time that the football is in the air.

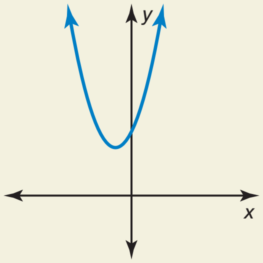
73. **CRITICAL THINKING** The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. Find the mean of the solutions. How is the mean of the solutions related to the graph of $y = ax^2 + bx + c$? Explain.

74. **HOW DO YOU SEE IT?** Match each graph with its discriminant. Explain your reasoning.

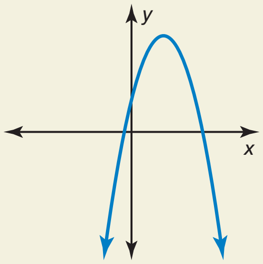
A.



B.



C.



a. $b^2 - 4ac > 0$

b. $b^2 - 4ac = 0$

c. $b^2 - 4ac < 0$

75. **CRITICAL THINKING** You are trying to hang a tire swing. To get the rope over a tree branch that is 15 feet high, you tie the rope to a weight and throw it over the branch. You release the weight at a height s_0 of 5.5 feet. What is the minimum initial vertical velocity v_0 needed to reach the branch? (*Hint*: Use the equation $h = -16t^2 + v_0t + s_0$.)

76. **THOUGHT PROVOKING** Consider the graph of the standard form of a quadratic function $y = ax^2 + bx + c$. Then consider the Quadratic Formula as given by

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Write a graphical interpretation of the two parts of this formula.

77. **ANALYZING RELATIONSHIPS** Find the sum and product of $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. Then write a quadratic equation whose solutions have a sum of 2 and a product of $\frac{1}{2}$.

78. **WRITING A FORMULA** Derive a formula that can be used to find solutions of equations that have the form $ax^2 + x + c = 0$. Use your formula to solve $-2x^2 + x + 8 = 0$.

79. **MULTIPLE REPRESENTATIONS** If p is a solution of a quadratic equation $ax^2 + bx + c = 0$, then $(x - p)$ is a factor of $ax^2 + bx + c$.

a. Copy and complete the table for each pair of solutions.

Solutions	Factors	Quadratic equation
3, 4	$(x - 3), (x - 4)$	$x^2 - 7x + 12 = 0$
$-1, 6$		
0, 2		
$-\frac{1}{2}, 5$		

b. Graph the related function for each equation. Identify the zeros of the function.

CRITICAL THINKING **In Exercises 80–82, find all values of k for which the equation has (a) two solutions, (b) one solution, and (c) no solutions.**

80. $2x^2 + x + 3k = 0$

81. $x^2 - 4kx + 36 = 0$

82. $kx^2 + 5x - 16 = 0$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Find the sum or difference. *(Section 2.1)*

83. $(x^2 + 2) + (2x^2 - x)$

84. $(x^3 + x^2 - 4) + (3x^2 + 10)$

85. $(-2x + 1) - (-3x^2 + x)$

86. $(-3x^3 + x^2 - 12x) - (-6x^2 + 3x - 9)$

Find the product. *(Section 2.2 and Section 2.3)*

87. $(x + 2)(x - 2)$

88. $2x(3 - x + 5x^2)$

89. $(7 - x)(x - 1)$

4.1–4.5 What Did You Learn?

Core Vocabulary

counterexample, *p. 191*
radical expression, *p. 192*
simplest form of a radical, *p. 192*
rationalizing the denominator, *p.* 194
conjugates, *p. 194*
like radicals, *p. 196*
quadratic equation, *p. 202*
completing the square, *p. 216*
Quadratic Formula, *p. 226*
discriminant, *p. 228*

Core Concepts

Section 4.1

Product Property of Square Roots, *p. 192*
Quotient Property of Square Roots, *p. 192*
Rationalizing the Denominator, *p. 194*
Performing Operations with Radicals, *p. 196*

Section 4.2

Solving Quadratic Equations by Graphing, *p. 202*
Number of Solutions of a Quadratic Equation, *p. 203*
Finding Zeros of Functions, *p. 204*

Section 4.3

Solutions of $x^2 = d$, *p. 210*
Approximating Solutions of Quadratic Equations, *p. 211*

Section 4.4

Completing the Square, *p. 216*
Finding and Using Maximum and Minimum Values, *p. 218*

Section 4.5

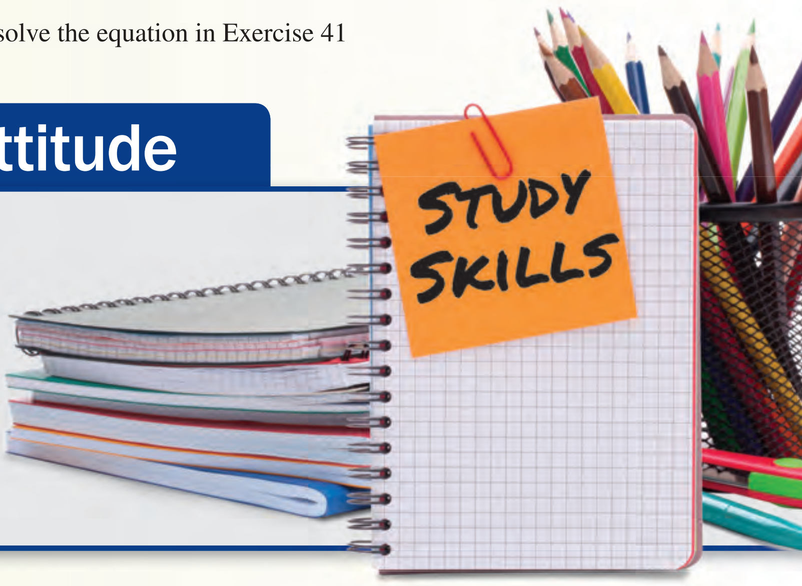
Quadratic Formula, *p. 226*
Interpreting the Discriminant, *p. 228*

Mathematical Practices

1. For each part of Exercise 100 on page 200 that is *sometimes* true, list all examples and counterexamples from the table that represent the sum or product being described.
2. Describe how solving a simpler equation can help you solve the equation in Exercise 41 on page 214.

Keeping a Positive Attitude

Do you ever feel frustrated or overwhelmed by math? You're not alone. Just take a deep breath and assess the situation. Try to find a productive study environment, review your notes and the examples in the textbook, and ask your teacher or friends for help.



4.1–4.5 Quiz

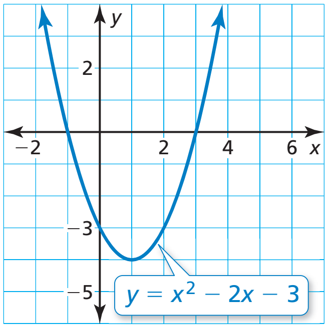
Simplify the expression. *(Section 4.1)*

1. $\sqrt{112x^3}$ **2.** $\sqrt{\frac{18}{81}}$ **3.** $\sqrt[3]{-625}$ **4.** $\frac{4}{\sqrt{11}}$

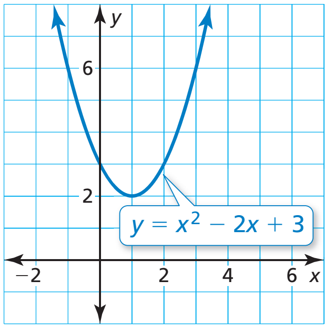
5. $\sqrt[3]{\frac{54x^4}{343y^6}}$ **6.** $\frac{6}{5+\sqrt{3}}$ **7.** $2\sqrt{5} + 7\sqrt{10} - 3\sqrt{20}$ **8.** $\sqrt{6}\left(7\sqrt{12} - 4\sqrt{3}\right)$

Use the graph to solve the equation. *(Section 4.2)*

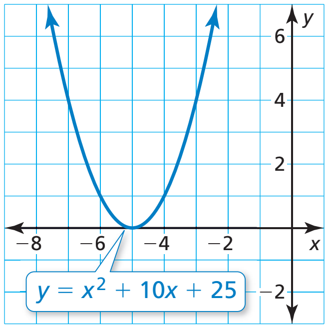
9. $x^2 - 2x - 3 = 0$



10. $x^2 - 2x + 3 = 0$



11. $x^2 + 10x + 25 = 0$



Solve the equation using square roots. *(Section 4.3)*

12. $4x^2 = 64$ **13.** $-3x^2 + 6 = 10$ **14.** $(x - 8)^2 = 1$

Solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary. *(Section 4.4)*

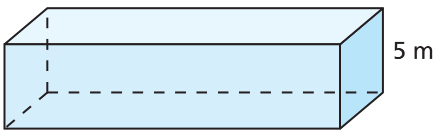
15. $x^2 - 6x + 8 = 0$ **16.** $x^2 + 12x + 4 = 0$ **17.** $4x(x + 6) = -40$

Solve the equation using the Quadratic Formula. Round your solutions to the nearest tenth, if necessary. *(Section 4.5)*

18. $x^2 + 9x + 14 = 0$ **19.** $x^2 + 4x = 1$ **20.** $-2x^2 + 7 = 3x$

21. Which method would you use to solve $2x^2 + 3x + 1 = 0$? Explain your reasoning. *(Section 4.5)*

22. The length of a rectangular prism is four times its width. The volume of the prism is 380 cubic meters. Find the length and width of the prism. *(Section 4.3)*



23. You cast a fishing lure into the water from a height of 4 feet above the water. The height h (in feet) of the fishing lure after t seconds can be modeled by the equation $h = -16t^2 + 24t + 4$. *(Section 4.5)*

a. After how many seconds does the fishing lure reach a height of 12 feet?

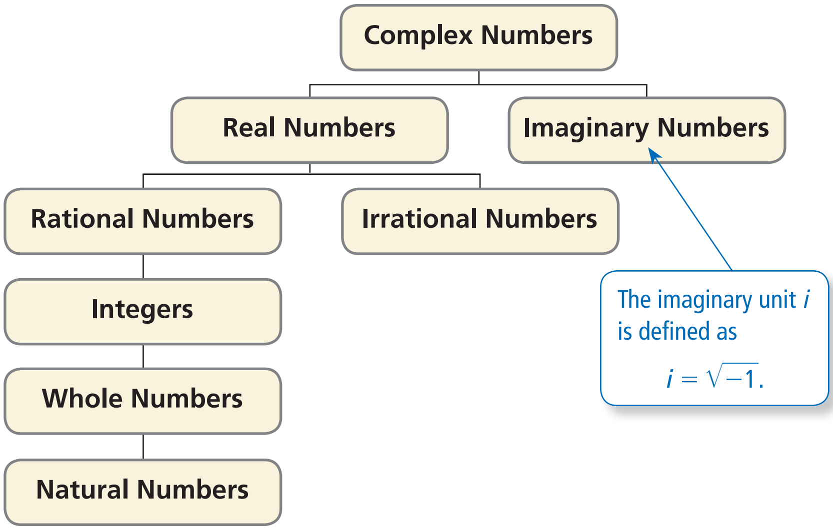
b. After how many seconds does the fishing lure hit the water?

4.6 Complex Numbers

Essential Question

What are the subsets of the set of complex numbers?

In your study of mathematics, you have probably worked with only *real numbers*, which can be represented graphically on the real number line. In this lesson, the system of numbers is expanded to include *imaginary numbers*. The real numbers and imaginary numbers compose the set of *complex numbers*.



EXPLORATION 1 Classifying Numbers

Work with a partner. Determine which subsets of the set of complex numbers contain each number.

a. $\sqrt{9}$ **b.** $\sqrt{0}$ **c.** $-\sqrt{4}$

d. $\sqrt{\frac{4}{9}}$ **e.** $\sqrt{2}$ **f.** $\sqrt{-1}$

ATTENDING TO PRECISION

To be proficient in math, you need to use clear definitions in your reasoning and discussions with others.

EXPLORATION 2 Simplifying i^2

Work with a partner. Justify each step in the simplification of i^2.

Algebraic Step	Justification
$i^2 = \left(\sqrt{-1}\right)^2$	
$= -1$	

Communicate Your Answer

3. What are the subsets of the set of complex numbers? Give an example of a number in each subset.

4. Is it possible for a number to be both whole and natural? natural and rational? rational and irrational? real and imaginary? Explain your reasoning.

5. Your friend claims that the conclusion in Exploration 2 is incorrect because $i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1(-1)} = \sqrt{1} = 1$. Is your friend correct? Explain.

4.6 Lesson

Core Vocabulary

imaginary unit *i*, *p. 238*
complex number, *p. 238*
imaginary number, *p. 238*
pure imaginary number, *p. 238*
complex conjugates, *p. 241*

What You Will Learn

- Define and use the imaginary unit *i*.
- Add and subtract complex numbers.
- Multiply complex numbers.

The Imaginary Unit *i*

Not all quadratic equations have real-number solutions. For example, $x^2 = -3$ has no real-number solutions because the square of any real number is never a negative number.

To overcome this problem, mathematicians created an expanded system of numbers using the **imaginary unit *i***, defined as $i = \sqrt{-1}$. Note that $i^2 = -1$. The imaginary unit i can be used to write the square root of *any* negative number.

Core Concept

The Square Root of a Negative Number

Property	Example
1. If r is a positive real number, then $\sqrt{-r} = i\sqrt{r}$.	$\sqrt{-3} = i\sqrt{3}$
2. By the first property, it follows that $(i\sqrt{r})^2 = -r$.	$(i\sqrt{3})^2 = i^2 \cdot 3 = -3$

EXAMPLE 1 Finding Square Roots of Negative Numbers

Find the square root of each number.

a. $\sqrt{-25}$ **b.** $\sqrt{-72}$ **c.** $-5\sqrt{-9}$

SOLUTION

a. $\sqrt{-25} = \sqrt{25} \cdot \sqrt{-1} = 5i$

b. $\sqrt{-72} = \sqrt{72} \cdot \sqrt{-1} = \sqrt{36} \cdot \sqrt{2} \cdot i = 6\sqrt{2}\, i = 6i\sqrt{2}$

c. $-5\sqrt{-9} = -5\sqrt{9} \cdot \sqrt{-1} = -5 \cdot 3 \cdot i = -15i$

Monitoring Progress

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Find the square root of the number.

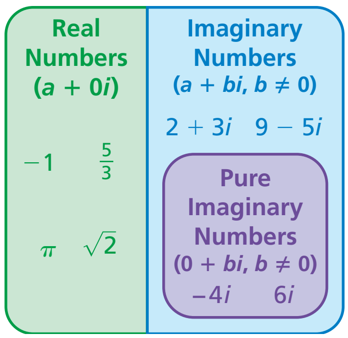
1. $\sqrt{-4}$ **2.** $\sqrt{-12}$ **3.** $-\sqrt{-36}$ **4.** $2\sqrt{-54}$

A **complex number** written in *standard form* is a number $a + bi$ where a and b are real numbers. The number a is the *real part*, and the number bi is the *imaginary part*.

$$a + bi$$

If $b \neq 0$, then $a + bi$ is an **imaginary number**. If $a = 0$ and $b \neq 0$, then $a + bi$ is a **pure imaginary number**. The diagram shows how different types of complex numbers are related.

Complex Numbers ($a + bi$)



Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.

EXAMPLE 2 Equality of Two Complex Numbers

Find the values of x and y that satisfy the equation $2x - 7i = 10 + yi$.

SOLUTION

Set the real parts equal to each other and the imaginary parts equal to each other.

$2x = 10$	Equate the real parts.	$-7i = yi$	Equate the imaginary parts.
$x = 5$	Solve for x.	$-7 = y$	Solve for y.

▶ So, $x = 5$ and $y = -7$.

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Find the values of x and y that satisfy the equation.

5. $x + 3i = 9 - yi$

6. $9 + 4yi = -2x + 3i$

Adding and Subtracting Complex Numbers

Core Concept

Sums and Differences of Complex Numbers

To add (or subtract) two complex numbers, add (or subtract) their real parts and their imaginary parts separately.

Sum of complex numbers: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference of complex numbers: $(a + bi) - (c + di) = (a - c) + (b - d)i$

EXAMPLE 3 Adding and Subtracting Complex Numbers

Add or subtract. Write the answer in standard form.

a. $(8 - i) + (5 + 4i)$

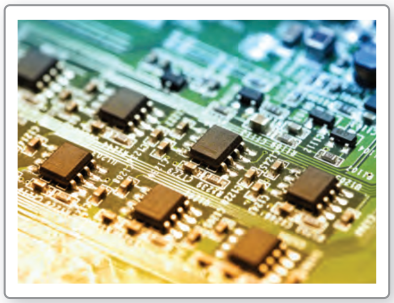
b. $(7 - 6i) - (3 - 6i)$

c. $13 - (2 + 7i) + 5i$

SOLUTION

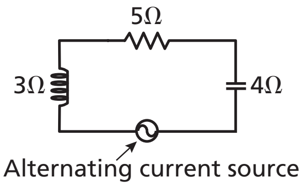
a. $(8 - i) + (5 + 4i) = (8 + 5) + (-1 + 4)i$	Definition of complex addition
$= 13 + 3i$	Write in standard form.
b. $(7 - 6i) - (3 - 6i) = (7 - 3) + (-6 + 6)i$	Definition of complex subtraction
$= 4 + 0i$	Simplify.
$= 4$	Write in standard form.
c. $13 - (2 + 7i) + 5i = [(13 - 2) - 7i] + 5i$	Definition of complex subtraction
$= (11 - 7i) + 5i$	Simplify.
$= 11 + (-7 + 5)i$	Definition of complex addition
$= 11 - 2i$	Write in standard form.

EXAMPLE 4 Solving a Real-Life Problem



Electrical circuit components, such as resistors, inductors, and capacitors, all oppose the flow of current. This opposition is called *resistance* for resistors and *reactance* for inductors and capacitors. Each of these quantities is measured in ohms. The symbol used for ohms is Ω, the uppercase Greek letter omega.

Component and symbol	Resistor	Inductor	Capacitor
Resistance or reactance (in ohms)	R	L	C
Impedance (in ohms)	R	Li	$-Ci$



The table shows the relationship between a component's resistance or reactance and its contribution to impedance. A *series circuit* is also shown with the resistance or reactance of each component labeled. The impedance for a series circuit is the sum of the impedances for the individual components. Find the impedance of the circuit.

SOLUTION

The resistor has a resistance of 5 ohms, so its impedance is 5 ohms. The inductor has a reactance of 3 ohms, so its impedance is $3i$ ohms. The capacitor has a reactance of 4 ohms, so its impedance is $-4i$ ohms.

$$\text{Impedance of circuit} = 5 + 3i + (-4i) = 5 - i$$

▶ The impedance of the circuit is $(5 - i)$ ohms.

Monitoring Progress

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Add or subtract. Write the answer in standard form.

7. $(9 - i) + (-6 + 7i)$

8. $(3 + 7i) - (8 - 2i)$

9. $-4 - (1 + i) - (5 + 9i)$

10. $5 + (-9 + 3i) + 6i$

11. WHAT IF? In Example 4, what is the impedance of the circuit when the capacitor is replaced with one having a reactance of 7 ohms?

Multiplying Complex Numbers

Many properties of real numbers are also valid for complex numbers, such as those used below to multiply two complex numbers.

$$(a + bi)(c + di) = a(c + di) + bi(c + di) \quad \text{Distributive Property}$$
$$= ac + (ad)i + (bc)i + (bd)i^2 \quad \text{Distributive Property}$$
$$= ac + (ad)i + (bc)i + (bd)(-1) \quad \text{Use } i^2 = -1.$$
$$= ac - bd + (ad)i + (bc)i \quad \text{Commutative Property}$$
$$= (ac - bd) + (ad + bc)i \quad \text{Associative Property}$$

This result shows a pattern that you could use to multiply two complex numbers. However, it is often more practical to use the Distributive Property or the FOIL Method, just as you do when multiplying real numbers or algebraic expressions.

EXAMPLE 5 Multiplying Complex Numbers

Multiply. Write the answer in standard form.

a. $4i(-6 + i)$ **b.** $(9 - 2i)(-4 + 7i)$

STUDY TIP

When simplifying an expression that involves complex numbers, be sure to simplify i^2 as -1.

SOLUTION

a. $4i(-6 + i) = -24i + 4i^2$ Distributive Property

$= -24i + 4(-1)$ Use $i^2 = -1$.

$= -4 - 24i$ Write in standard form.

b. $(9 - 2i)(-4 + 7i) = -36 + 63i + 8i - 14i^2$ Multiply using FOIL.

$= -36 + 71i - 14(-1)$ Simplify and use $i^2 = -1$.

$= -36 + 71i + 14$ Simplify.

$= -22 + 71i$ Write in standard form.

Monitoring Progress

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Multiply. Write the answer in standard form.

12. $(-3i)(10i)$ **13.** $i(8 - i)$ **14.** $(3 + i)(5 - i)$

Pairs of complex numbers of the forms $a + bi$ and $a - bi$, where $b \neq 0$, are called **complex conjugates**. Consider the product of complex conjugates below.

$(a + bi)(a - bi) = a^2 - (ab)i + (ab)i - b^2i^2$ Multiply using FOIL.

$= a^2 - b^2(-1)$ Simplify and use $i^2 = -1$.

$= a^2 + b^2$ Simplify.

Because a and b are real numbers, $a^2 + b^2$ is a real number. So, the product of complex conjugates is a real number.

LOOKING FOR STRUCTURE

You can use the pattern $(a + bi)(a - bi) = a^2 + b^2$, where $a = 5$ and $b = 2$, to find the product in Example 6.

$(5 + 2i)(5 - 2i) = 5^2 + 2^2$

$= 25 + 4$

$= 29$

EXAMPLE 6 Multiplying Complex Conjugates

Multiply $5 + 2i$ by its complex conjugate.

SOLUTION

The complex conjugate of $5 + 2i$ is $5 - 2i$.

$(5 + 2i)(5 - 2i) = 25 - 10i + 10i - 4i^2$ Multiply using FOIL.

$= 25 - 4(-1)$ Simplify and use $i^2 = -1$.

$= 29$ Simplify.

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Multiply the complex number by its complex conjugate.

15. $1 + i$ **16.** $4 - 7i$ **17.** $-3 - 2i$

4.6 Exercises

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Vocabulary and Core Concept Check

1. **VOCABULARY** What is the imaginary unit i defined as and how can you use i?

2. **COMPLETE THE SENTENCE** For the complex number $5 + 2i$, the imaginary part is ____ and the real part is ____.

3. **WRITING** Describe how to add complex numbers.

4. **WHICH ONE DOESN'T BELONG?** Which number does *not* belong with the other three? Explain your reasoning.

   $3 + 0i$ $\quad$ $2 + 5i$ $\quad$ $\sqrt{3} + 6i$ $\quad$ $0 - 7i$

Monitoring Progress and Modeling with Mathematics

In Exercises 5–12, find the square root of the number. *(See Example 1.)*

5. $\sqrt{-36}$
6. $\sqrt{-64}$
7. $\sqrt{-18}$
8. $\sqrt{-24}$
9. $2\sqrt{-16}$
10. $-3\sqrt{-49}$
11. $-4\sqrt{-32}$
12. $6\sqrt{-63}$

In Exercises 13–20, find the values of x and y that satisfy the equation. *(See Example 2.)*

13. $4x + 2i = 8 + yi$
14. $3x + 6i = 27 + yi$
15. $-10x + 12i = 20 + 3yi$
16. $9x - 18i = -36 + 6yi$
17. $2x - yi = 14 + 12i$
18. $-12x + yi = 60 - 13i$
19. $54 - \frac{1}{7}yi = 9x - 4i$
20. $15 - 3yi = \frac{1}{2}x + 2i$

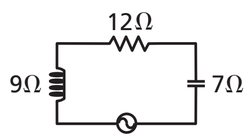
In Exercises 21–30, add or subtract. Write the answer in standard form. *(See Example 3.)*

21. $(6 - i) + (7 + 3i)$
22. $(9 + 5i) + (11 + 2i)$
23. $(12 + 4i) - (3 - 7i)$
24. $(2 - 15i) - (4 + 5i)$
25. $(12 - 3i) + (7 + 3i)$
26. $(16 - 9i) - (2 - 9i)$
27. $7 - (3 + 4i) + 6i$
28. $16 - (2 - 3i) - i$
29. $-10 + (6 - 5i) - 9i$
30. $-3 + (8 + 2i) + 7i$

31. **USING STRUCTURE** Write each expression as a complex number in standard form.
    a. $\sqrt{-9} + \sqrt{-4} - \sqrt{16}$
    b. $\sqrt{-16} + \sqrt{8} + \sqrt{-36}$

32. **REASONING** The additive inverse of a complex number z is a complex number z_a such that $z + z_a = 0$. Find the additive inverse of each complex number.
    a. $z = 1 + i$ $\quad$ b. $z = 3 - i$ $\quad$ c. $z = -2 + 8i$

In Exercises 33–36, find the impedance of the series circuit. *(See Example 4.)*

33.


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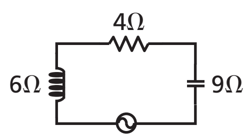

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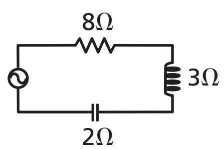

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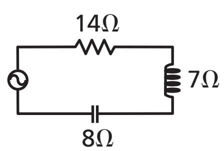


In Exercises 37–44, multiply. Write the answer in standard form. *(See Example 5.)*

37. $3i(-5 + i)$

38. $2i(7 - i)$

39. $(3 - 2i)(4 + i)$

40. $(7 + 5i)(8 - 6i)$

41. $(5 - 2i)(-2 - 3i)$

42. $(-1 + 8i)(9 + 3i)$

43. $(3 - 6i)^2$

44. $(8 + 3i)^2$

JUSTIFYING STEPS In Exercises 45 and 46, justify each step in performing the operation.

45. $11 - (4 + 3i) + 5i$

$= [(11 - 4) - 3i] + 5i$

$= (7 - 3i) + 5i$

$= 7 + (-3 + 5)i$

$= 7 + 2i$

46. $(3 + 2i)(7 - 4i)$

$= 21 - 12i + 14i - 8i^2$

$= 21 + 2i - 8(-1)$

$= 21 + 2i + 8$

$= 29 + 2i$

REASONING In Exercises 47 and 48, place the tiles in the expression to make a true statement.

47. $(\_\_\_\_ - \_\_\_\_ i) - (\_\_\_\_ - \_\_\_\_ i) = 2 - 4i$

7 4 3 6

48. $\_\_\_\_ i(\_\_\_\_ + \_\_\_\_ i) = -18 - 10i$

−5 9 2

In Exercises 49–54, multiply the complex number by its complex conjugate. *(See Example 6.)*

49. $1 - i$

50. $8 + i$

51. $4 + 2i$

52. $5 - 6i$

53. $-2 + 2i$

54. $-1 - 9i$

55. **OPEN-ENDED** Write a pair of imaginary numbers whose product is 80.

56. **NUMBER SENSE** Write the complex conjugate of $1 - \sqrt{-12}$. Then find the product of the complex conjugates.

57. **USING STRUCTURE** Expand $(a - bi)^2$ and write the result in standard form. Use your result to check your answer in Exercise 43.

58. **USING STRUCTURE** Expand $(a + bi)^2$ and write the result in standard form. Use your result to check your answer in Exercise 44.

ERROR ANALYSIS In Exercises 59 and 60, describe and correct the error in performing the operation and writing the answer in standard form.

59.

$$\begin{aligned}(3 + 2i)(5 - i) &= 15 - 3i + 10i - 2i^2 \\ &= 15 + 7i - 2i^2 \\ &= -2i^2 + 7i + 15\end{aligned}$$

60.

$$\begin{aligned}(4 + 6i)^2 &= (4)^2 + (6i)^2 \\ &= 16 + 36i^2 \\ &= 16 + (36)(-1) \\ &= -20\end{aligned}$$

61. **NUMBER SENSE** Simplify each expression. Then classify your results in the table below.

a. $(-4 + 7i) + (-4 - 7i)$

b. $(2 - 6i) - (-10 + 4i)$

c. $(25 + 15i) - (25 - 6i)$

d. $(5 + i)(8 - i)$

e. $(17 - 3i) + (-17 - 6i)$

f. $(-1 + 2i)(11 - i)$

g. $(7 + 5i) + (7 - 5i)$

h. $(-3 + 6i) - (-3 - 8i)$

Real numbers	Imaginary numbers	Pure imaginary numbers

62. **MAKING AN ARGUMENT** The Product Property of Square Roots states $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$. Your friend concludes $\sqrt{-4} \cdot \sqrt{-9} = \sqrt{36} = 6$. Is your friend correct? Explain.

63. **FINDING A PATTERN** Make a table that shows the powers of i from i^1 to i^8 in the first row and the simplified forms of these powers in the second row. Describe the pattern you observe in the table. Verify the pattern continues by evaluating the next four powers of i.

64. **HOW DO YOU SEE IT?** The coordinate system shown below is called the *complex plane*. In the complex plane, the point that corresponds to the complex number $a + bi$ is (a, b). Match each complex number with its corresponding point.

a. 2

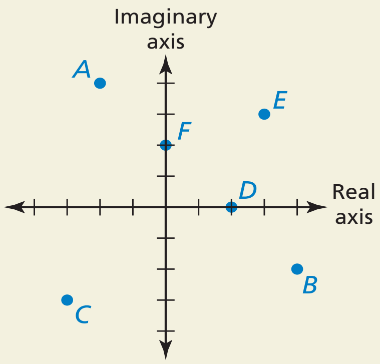
b. $2i$

c. $4 - 2i$

d. $3 + 3i$

e. $-2 + 4i$

f. $-3 - 3i$



In Exercises 65–70, write the expression as a complex number in standard form.

65. $(3 + 4i) - (7 - 5i) + 2i(9 + 12i)$

66. $3i(2 + 5i) + (6 - 7i) - (9 + i)$

67. $(3 + 5i)(2 - 7i^4)$

68. $2i^3(5 - 12i)$

69. $(2 + 4i^5) + (1 - 9i^6) - (3 + i^7)$

70. $(8 - 2i^4) + (3 - 7i^8) - (4 + i^9)$

71. **NUMBER SENSE** Write a pair of complex numbers whose sum is -4 and whose product is 53.

72. **COMPARING METHODS** Describe the two different methods shown for writing the complex expression in standard form. Which method do you prefer? Explain.

Method 1

$$\begin{aligned}4i(2-3i)+4i(1-2i) &= 8i-12i^2+4i-8i^2\\ &= 8i-12(-1)+4i-8(-1)\\ &= 20+12i\end{aligned}$$

Method 2

$$\begin{aligned}4i(2-3i)+4i(1-2i) &= 4i[(2-3i)+(1-2i)]\\ &= 4i[3-5i]\\ &= 12i-20i^2\\ &= 12i-20(-1)\\ &= 20+12i\end{aligned}$$

73. **CRITICAL THINKING** **Determine whether each statement is *true* or *false*. If it is true, give an example. If it is false, give a counterexample.**

a. The sum of two imaginary numbers is an imaginary number.

b. The product of two pure imaginary numbers is a real number.

c. A pure imaginary number is an imaginary number.

d. A complex number is a real number.

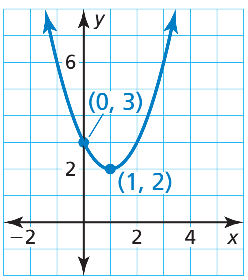
74. **THOUGHT PROVOKING** Create a circuit that has an impedance of $14 - 3i$.

Maintaining Mathematical Proficiency

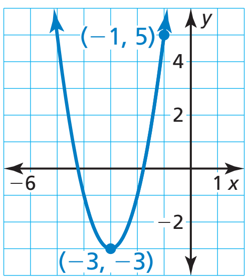
Reviewing what you learned in previous grades and lessons

Write a quadratic function in vertex form whose graph is shown. *(Section 3.4)*

75.



76.



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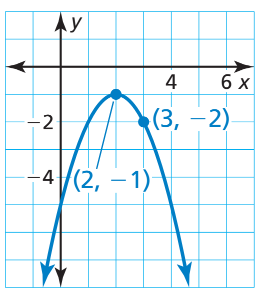


Find the value of c that completes the square. *(Section 4.4)*

78. $x^2 + 8x + c$

79. $x^2 - 20x + c$

80. $x^2 - 7x + c$

4.7 Solving Quadratic Equations with Complex Solutions

Essential Question

How can you determine whether a quadratic equation has real solutions or imaginary solutions?

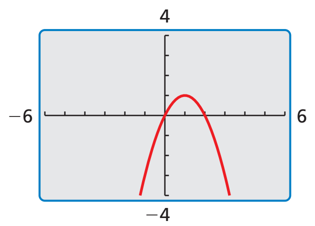
EXPLORATION 1 Using Graphs to Solve Quadratic Equations

Work with a partner. Use the discriminant of $f(x) = 0$ and the sign of the leading coefficient of $f(x)$ to match each quadratic function with its graph. Explain your reasoning. Then find the real solution(s) (if any) of each quadratic equation $f(x) = 0$.

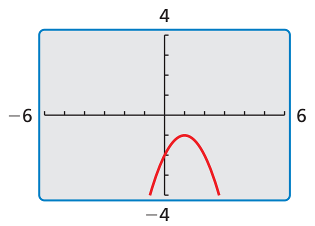
a. $f(x) = x^2 - 2x$ **b.** $f(x) = x^2 - 2x + 1$ **c.** $f(x) = x^2 - 2x + 2$

d. $f(x) = -x^2 + 2x$ **e.** $f(x) = -x^2 + 2x - 1$ **f.** $f(x) = -x^2 + 2x - 2$

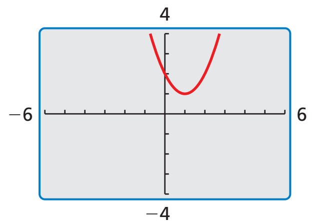
A.



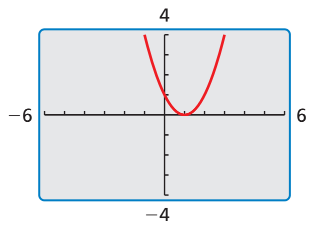
B.



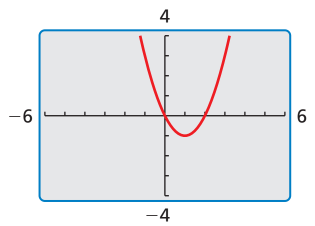
C.



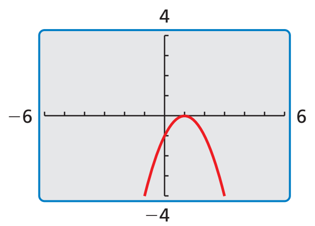
D.



E.



F.



EXPLORATION 2 Finding Imaginary Solutions

Work with a partner. What do you know about the discriminants of quadratic equations that have no real solutions? Use the Quadratic Formula and what you learned about the imaginary unit i to find the *imaginary* solutions of each equation in Exploration 1 that has no real solutions. Use substitution to check your answers.

MAKING SENSE OF PROBLEMS

To be proficient in math, you need to make conjectures about the form and meaning of solutions.

Communicate Your Answer

3. How can you determine whether a quadratic equation has real solutions or imaginary solutions?

4. Describe the number and type of solutions of $x^2 + 2x + 3 = 0$. How do you know? What are the solutions?

4.7 Lesson

What You Will Learn

- Solve quadratic equations and find zeros of quadratic functions.
- Use the discriminant.

Finding Solutions and Zeros

Previously, you learned that you can use the discriminant of a quadratic equation to determine whether the equation has two real solutions, one real solution, or no real solutions. When the discriminant is negative, you can use the imaginary unit i to write two *imaginary* solutions of the equation. So, all quadratic equations have complex number solutions.

You have solved quadratic equations with real solutions. Now you will solve quadratic equations with *imaginary* solutions.

STUDY TIP

In general, every polynomial equation has complex number solutions. This is implied by the *Fundamental Theorem of Algebra*. You will learn more about this theorem in a future course.

EXAMPLE 1 **Solving Quadratic Equations**

Solve each equation.

a. $x^2 + 9 = 0$ **b.** $x^2 + 4x + 5 = 0$ **c.** $5x^2 - 4x + 1 = 0$

SOLUTION

a. The equation does not have an x-term. So, solve using square roots.

$x^2 + 9 = 0$ Write the equation.

$x^2 = -9$ Subtract 9 from each side.

$x = \pm\sqrt{-9}$ Take the square root of each side.

$x = \pm 3i$ Write in terms of i.

LOOKING FOR STRUCTURE

You can use the pattern $(a + bi)(a - bi) = a^2 + b^2$ to rewrite $x^2 + 9 = 0$ as $(x + 3i)(x - 3i) = 0$. So, $x = \pm 3i$.

b. The coefficient of the x^2-term is 1, and the coefficient of the x-term is an even number. So, solve by completing the square.

$$x^2 + 4x + 5 = 0$$
$$x^2 + 4x = -5$$
$$x^2 + 4x + 4 = -5 + 4$$
$$(x + 2)^2 = -1$$
$$x + 2 = \pm\sqrt{-1}$$
$$x = -2 \pm \sqrt{-1}$$
$$x = -2 \pm i$$

Check You can check imaginary solutions algebraically. The check for one of the imaginary solutions, $-2 + i$, is shown.

$$(-2 + i)^2 + 4(-2 + i) + 5 \stackrel{?}{=} 0$$
$$3 - 4i - 8 + 4i + 5 \stackrel{?}{=} 0$$
$$0 = 0 \checkmark$$

c. The equation is not factorable, and completing the square would result in fractions. So, solve using the Quadratic Formula.

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(1)}}{2(5)}$ Substitute 5 for a, -4 for b, and 1 for c.

$x = \dfrac{4 \pm \sqrt{-4}}{10}$ Simplify.

$x = \dfrac{4 \pm 2i}{10}$ Write in terms of i.

$x = \dfrac{2 \pm i}{5}$ Simplify.

EXAMPLE 2 Finding Zeros of a Quadratic Function

Find the zeros of $f(x) = 4x^2 + 20$.

SOLUTION

$4x^2 + 20 = 0$ — Set $f(x)$ equal to 0.

$4x^2 = -20$ — Subtract 20 from each side.

$x^2 = -5$ — Divide each side by 4.

$x = \pm\sqrt{-5}$ — Take the square root of each side.

$x = \pm i\sqrt{5}$ — Write in terms of i.

▶ So, the zeros of f are $i\sqrt{5}$ and $-i\sqrt{5}$.

Check

$f(i\sqrt{5}) = 4(i\sqrt{5})^2 + 20$
$= 4(-5) + 20$
$= 0$ ✓

$f(-i\sqrt{5}) = 4(-i\sqrt{5})^2 + 20$
$= 4(-5) + 20$
$= 0$ ✓

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Solve the equation using any method. Explain your choice of method.

1. $-x^2 - 25 = 0$ **2.** $x^2 - 4x + 8 = 0$ **3.** $8x^2 + 5 = 12x$

Find the zeros of the function.

4. $f(x) = -2x^2 - 18$ **5.** $f(x) = 9x^2 + 1$ **6.** $f(x) = x^2 - 6x + 10$

Using the Discriminant

EXAMPLE 3 Writing an Equation

Find a possible pair of integer values for a and c so that the equation $ax^2 - 4x + c = 0$ has two imaginary solutions. Then write the equation.

SOLUTION

For the equation to have two imaginary solutions, the discriminant must be less than zero.

$b^2 - 4ac < 0$ — Write the discriminant.

$(-4)^2 - 4ac < 0$ — Substitute -4 for b.

$16 - 4ac < 0$ — Evaluate the power.

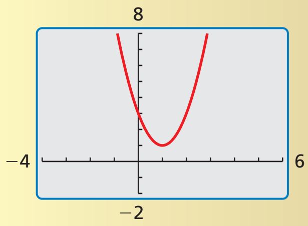
$-4ac < -16$ — Subtract 16 from each side.

$ac > 4$ — Divide each side by -4. Reverse inequality symbol.

Because $ac > 4$, choose two integers whose product is greater than 4, such as $a = 2$ and $c = 3$.

▶ So, one possible equation is $2x^2 - 4x + 3 = 0$.

Check The graph of $y = 2x^2 - 4x + 3$ does not have any x-intercepts. ✓



ANOTHER WAY

Another possible equation in Example 3 is $3x^2 - 4x + 2 = 0$. You can obtain this equation by letting $a = 3$ and $c = 2$.

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7. Find a possible pair of integer values for a and c so that the equation $ax^2 + 3x + c = 0$ has two imaginary solutions. Then write the equation.

The function $h = -16t^2 + s_0$ is used to model the height of a *dropped* object, where h is the height (in feet), t is the time in motion (in seconds), and s_0 is the initial height (in feet). For an object that is *launched* or *thrown*, an extra term v_0t must be added to the model to account for the object's initial vertical velocity v_0 (in feet per second).

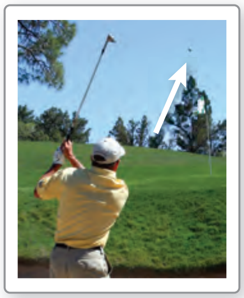
$h = -16t^2 + s_0$ — Object is dropped.

$h = -16t^2 + v_0t + s_0$ — Object is launched or thrown.

STUDY TIP

These models assume that the force of air resistance on the object is negligible. Also, these models apply only to objects on Earth. For planets with stronger or weaker gravitational forces, different models are used.

As shown below, the value of v_0 can be positive, negative, or zero depending on whether the object is launched upward, downward, or parallel to the ground.



$V_0 > 0$

$V_0 < 0$

$V_0 = 0$

EXAMPLE 4 Modeling a Launched Object

A juggler tosses a ball into the air. The ball leaves the juggler's hand 4 feet above the ground and has an initial vertical velocity of 30 feet per second. Does the ball reach a height of 25 feet? 10 feet? Explain your reasoning.

SOLUTION

Because the ball is *thrown*, use the model $h = -16t^2 + v_0t + s_0$ to write a function that represents the height of the ball.

$h = -16t^2 + v_0t + s_0$ — Write the height model.

$h = -16t^2 + 30t + 4$ — Substitute 30 for v_0 and 4 for s_0.

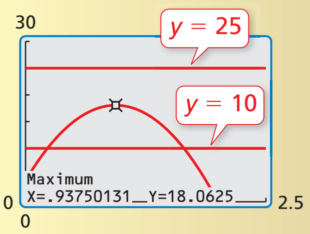
To determine whether the ball reaches each height, substitute each height for h to create two equations. Then solve each equation using the Quadratic Formula.

$25 = -16t^2 + 30t + 4$	$10 = -16t^2 + 30t + 4$
$0 = -16t^2 + 30t - 21$	$0 = -16t^2 + 30t - 6$
$t = \dfrac{-30 \pm \sqrt{30^2 - 4(-16)(-21)}}{2(-16)}$	$t = \dfrac{-30 \pm \sqrt{30^2 - 4(-16)(-6)}}{2(-16)}$
$t = \dfrac{-30 \pm \sqrt{-444}}{-32}$	$t = \dfrac{-30 \pm \sqrt{516}}{-32}$

When $h = 25$, the equation has two imaginary solutions because the discriminant is negative. When $h = 10$, the equation has two real solutions, $t \approx 0.23$ and $t \approx 1.65$.

▶ So, the ball reaches a height of 10 feet, but it does not reach a height of 25 feet.

Check The graph shows that the ball reaches a height of 10 feet but not 25 feet. ✓



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8. The ball leaves the juggler's hand with an initial vertical velocity of 40 feet per second. Does the ball reach a height of 30 feet? 20 feet? Explain.

4.7 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

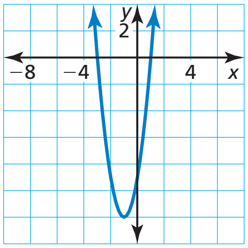
1. **COMPLETE THE SENTENCE** When the graph of a quadratic function $y = f(x)$ has no x-intercepts, the equation $f(x) = 0$ has two ____________ solutions.

2. **WRITING** Can a quadratic equation with real coefficients have one imaginary solution? Explain.

Monitoring Progress and Modeling with Mathematics

ANALYZING EQUATIONS **In Exercises 3–6, use the discriminant to match the quadratic equation with the graph of the related function. Then describe the number and type of solutions of the equation.**

3. $x^2 - 6x + 25 = 0$
4. $2x^2 - 20x + 50 = 0$
5. $3x^2 + 6x - 9 = 0$
6. $5x^2 - 10x - 35 = 0$

A.


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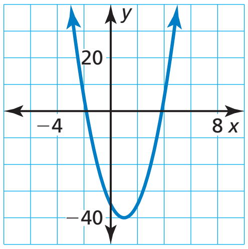

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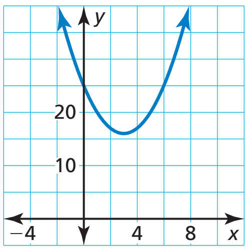
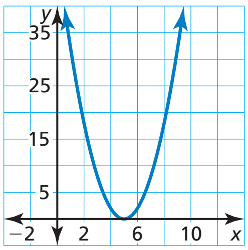


D.


In Exercises 7–20, solve the equation using any method. Explain your choice of method. *(See Example 1.)*

7. $x^2 + 49 = 0$
8. $2x^2 - 7 = -3$
9. $x^2 - 4x + 3 = 0$
10. $3x^2 + 6x + 3 = 0$
11. $x^2 + 6x + 15 = 0$
12. $6x^2 - 2x + 1 = 0$
13. $9x^2 + 17 = 24x$
14. $-3x = 2x^2 - 4$
15. $-10x = -25 - x^2$
16. $-2x^2 - 5 = -2x$
17. $-4x^2 + 3x = -5$
18. $3x^2 + 87 = 30x$
19. $-z^2 = -12z + 6$
20. $-7w + 6 = -4w^2$

21. **ERROR ANALYSIS** Describe and correct the error in solving the equation.

$$x^2 + 10x + 74 = 0$$

$$x = \frac{-10 \pm \sqrt{10^2 - 4(1)(74)}}{2(1)}$$

$$= \frac{-10 \pm \sqrt{-196}}{2}$$

$$= \frac{-10 \pm 14}{2}$$

$$= -12 \text{ or } 2$$

22. **REASONING** Write a quadratic equation in the form $ax^2 + bx + c = 0$ that has the solutions $x = 1 \pm i$.

In Exercises 23–28, find the zeros of the function. *(See Example 2.)*

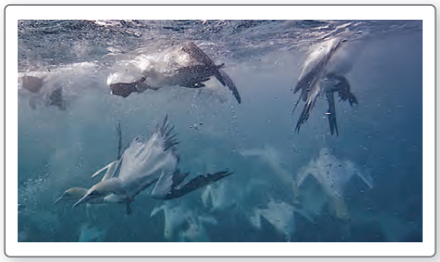
23. $f(x) = 5x^2 + 35$
24. $g(x) = -3x^2 + 24$
25. $h(x) = x^2 + 8x - 13$
26. $r(x) = 8x^2 + 4x + 5$
27. $m(x) = -5x^2 + 50x - 135$
28. $r(x) = 4x^2 + 9x + 3$

OPEN-ENDED **In Exercises 29–32, find a possible pair of integer values for a and c so that the quadratic equation has the given solution(s). Then write the equation.** *(See Example 3.)*

29. $ax^2 + 4x + c = 0$; two imaginary solutions
30. $ax^2 - 8x + c = 0$; two real solutions
31. $ax^2 + 10x = c$; one real solution
32. $-4x + c = -ax^2$; two imaginary solutions

MODELING WITH MATHEMATICS In Exercises 33 and 34, write a function that represents the situation.

33. A gannet is a bird that feeds on fish by diving into the water. A gannet spots a fish on the surface of the water and dives 100 feet to catch it. The bird plunges toward the water with an initial vertical velocity of −88 feet per second.



34. An archer is shooting at targets. The height of the arrow is 5 feet above the ground. Due to safety rules, the archer must aim the arrow parallel to the ground.

35. **MODELING WITH MATHEMATICS** A lacrosse player throws a ball in the air from an initial height of 7 feet. The ball has an initial vertical velocity of 35 feet per second. Does the ball reach a height of 30 feet? 26 feet? Explain your reasoning. *(See Example 4.)*

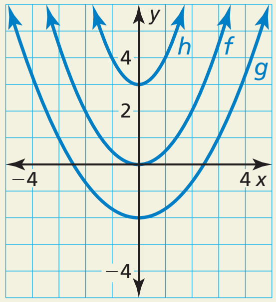
36. **PROBLEM SOLVING** A rocketry club is launching model rockets. The launching pad is 30 feet above the ground. Your model rocket has an initial vertical velocity of 105 feet per second. Your friend's model rocket has an initial vertical velocity of 100 feet per second.

a. Does your rocket reach a height of 200 feet? Does your friend's rocket? Explain your reasoning.

b. Which rocket is in the air longer? How much longer?

37. **CRITICAL THINKING** When a quadratic equation with real coefficients has imaginary solutions, why are the solutions complex conjugates? As part of your explanation, show that there is no such equation with solutions of $3i$ and $-2i$.

38. **HOW DO YOU SEE IT?** The graphs of three functions are shown. Which function(s) has real zeros? imaginary zeros? Explain your reasoning.



39. **USING STRUCTURE** Use the Quadratic Formula to write a quadratic equation that has the solutions $x = \dfrac{-8 \pm \sqrt{-176}}{-10}$.

40. **THOUGHT PROVOKING** Describe a real-life story that could be modeled by $h = -16t^2 + v_0t + s_0$. Write the height model for your story and determine how long your object is in the air.

41. **MODELING WITH MATHEMATICS** The Stratosphere Tower in Las Vegas is 921 feet tall and has a "needle" at its top that extends even higher into the air. A thrill ride called Big Shot catapults riders 160 feet up the needle and then lets them fall back to the launching pad.

a. The height h (in feet) of a rider on the Big Shot can be modeled by $h = -16t^2 + v_0t + 921$, where t is the elapsed time (in seconds) after launch and v_0 is the initial vertical velocity (in feet per second). Find v_0 using the fact that the maximum value of h is $921 + 160 = 1081$ feet.

b. A brochure for the Big Shot states that the ride up the needle takes 2 seconds. Compare this time to the time given by the model $h = -16t^2 + v_0t + 921$, where v_0 is the value you found in part (a). Discuss the accuracy of the model.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the system of linear equations using any method. Explain why you chose the method. *(Skills Review Handbook)*

42. $y = -x + 4$
$y = 2x - 8$

43. $x = 16 - 4y$
$3x + 4y = 8$

44. $2x - y = 7$
$2x + 7y = 31$

45. $3x - 2y = -20$
$x + 1.2y = 6.4$

Find (a) the axis of symmetry and (b) the vertex of the graph of the function. *(Section 3.3)*

46. $y = -x^2 + 2x + 1$

47. $y = 2x^2 - x + 3$

48. $f(x) = 0.5x^2 + 2x + 5$

4.8 Solving Nonlinear Systems of Equations

Essential Question How can you solve a system of two equations when one is linear and the other is quadratic?

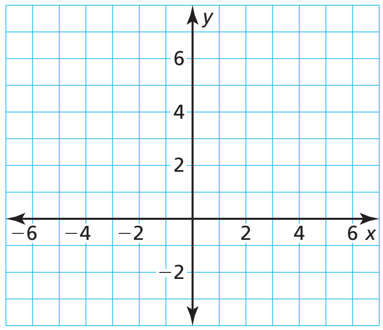
EXPLORATION 1 Solving a System of Equations

Work with a partner. Solve the system of equations by graphing each equation and finding the points of intersection.

System of Equations

$y = x + 2$ Linear

$y = x^2 + 2x$ Quadratic



EXPLORATION 2 Analyzing Systems of Equations

Work with a partner. Match each system of equations with its graph. Then solve the system of equations.

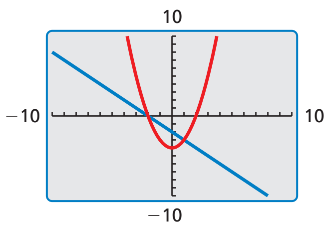
a. $y = x^2 - 4$
$y = -x - 2$

b. $y = x^2 - 2x + 2$
$y = 2x - 2$

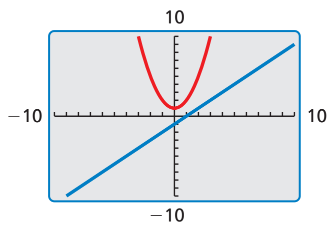
c. $y = x^2 + 1$
$y = x - 1$

d. $y = x^2 - x - 6$
$y = 2x - 2$

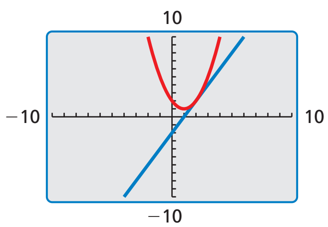
A.



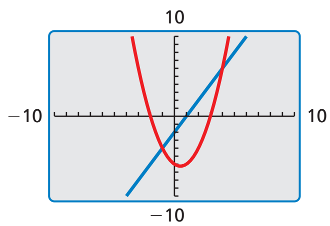
B.



C.



D.



MAKING SENSE OF PROBLEMS

To be proficient in math, you need to analyze givens, relationships, and goals.

Communicate Your Answer

3. How can you solve a system of two equations when one is linear and the other is quadratic?

4. Write a system of equations (one linear and one quadratic) that has (a) no solutions, (b) one solution, and (c) two solutions. Your systems should be different from those in Explorations 1 and 2.

4.8 Lesson

Core Vocabulary

system of nonlinear equations, *p. 252*

Previous

system of linear equations

What You Will Learn

- Solve systems of nonlinear equations by graphing.
- Solve systems of nonlinear equations algebraically.
- Approximate solutions of nonlinear systems and equations.

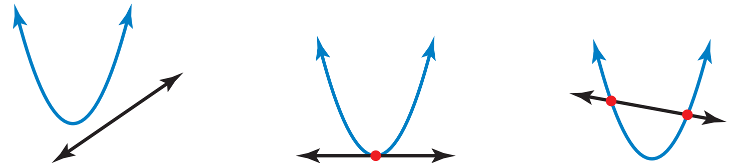
Solving Nonlinear Systems by Graphing

The methods for solving systems of linear equations can also be used to solve *systems of nonlinear equations*. A **system of nonlinear equations** is a system in which at least one of the equations is nonlinear.

When a nonlinear system consists of a linear equation and a quadratic equation, the graphs can intersect in zero, one, or two points. So, the system can have zero, one, or two solutions, as shown.

STUDY TIP

In this section, *solutions* of a nonlinear system refer to the *real* solutions of the system.



No solutions **One solution** **Two solutions**

EXAMPLE 1 Solving a Nonlinear System by Graphing

Solve the system by graphing.

$y = 2x^2 + 5x - 1$ Equation 1

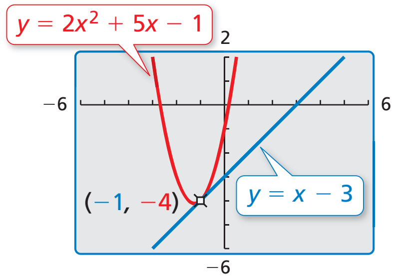
$y = x - 3$ Equation 2

SOLUTION

Step 1 Graph each equation.

Step 2 Estimate the point of intersection. The graphs appear to intersect at $(-1, -4)$.

Step 3 Check the point from Step 2 by substituting the coordinates into each of the original equations.



Equation 1

$$y = 2x^2 + 5x - 1$$
$$-4 \stackrel{?}{=} 2(-1)^2 + 5(-1) - 1$$
$$-4 = -4 \checkmark$$

Equation 2

$$y = x - 3$$
$$-4 \stackrel{?}{=} -1 - 3$$
$$-4 = -4 \checkmark$$

▶ The solution is $(-1, -4)$.

Monitoring Progress

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Solve the system by graphing.

1. $y = x^2 + 4x - 4$
$y = 2x - 5$

2. $y = -x + 6$
$y = -2x^2 - x + 3$

3. $y = 3x - 15$
$y = \frac{1}{2}x^2 - 2x - 7$

REMEMBER
The algebraic procedures that you use to solve nonlinear systems are similar to the procedures that you used to solve linear systems.

Solving Nonlinear Systems Algebraically

EXAMPLE 2 Solving a Nonlinear System by Substitution

Solve the system by substitution.

$y = x^2 + x - 1$ Equation 1

$y = -2x + 3$ Equation 2

SOLUTION

Step 1 The equations are already solved for y.

Step 2 Substitute $-2x + 3$ for y in Equation 1 and solve for x.

$-2x + 3 = x^2 + x - 1$ Substitute $-2x + 3$ for y in Equation 1.

$3 = x^2 + 3x - 1$ Add $2x$ to each side.

$0 = x^2 + 3x - 4$ Subtract 3 from each side.

$0 = (x + 4)(x - 1)$ Factor the polynomial.

$x + 4 = 0$ or $x - 1 = 0$ Zero-Product Property

$x = -4$ or $x = 1$ Solve for x.

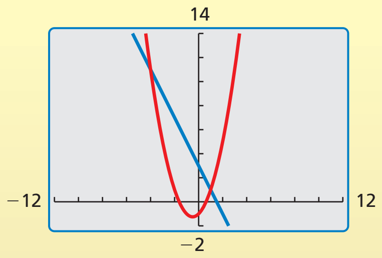
Step 3 Substitute -4 and 1 for x in Equation 2 and solve for y.

$y = -2(-4) + 3$ Substitute for x in Equation 2. $y = -2(1) + 3$

$= 11$ Simplify. $= 1$

▶ So, the solutions are $(-4, 11)$ and $(1, 1)$.

Check
Use a graphing calculator to check your answer. Notice that the graphs have two points of intersection at $(-4, 11)$ and $(1, 1)$.



EXAMPLE 3 Solving a Nonlinear System by Elimination

Solve the system by elimination.

$y = x^2 - 3x - 2$ Equation 1

$y = -3x - 8$ Equation 2

SOLUTION

Step 1 Because the coefficients of the y-terms are the same, you do not need to multiply either equation by a constant.

Step 2 Subtract Equation 2 from Equation 1.

$y = x^2 - 3x - 2$ Equation 1

$y = \quad -3x - 8$ Equation 2

$0 = x^2 \quad + 6$ Subtract the equations.

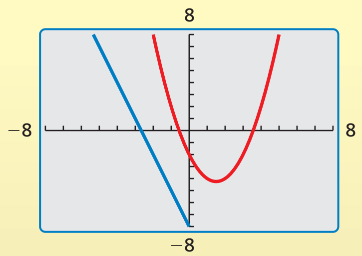
Step 3 Solve for x.

$0 = x^2 + 6$ Resulting equation from Step 2

$-6 = x^2$ Subtract 6 from each side.

▶ The square of a real number cannot be negative. So, the system has no real solutions.

Check
Use a graphing calculator to check your answer. The graphs do not intersect.



Monitoring Progress

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Solve the system by substitution.

4. $y = x^2 + 9$
$y = 9$

5. $y = -5x$
$y = x^2 - 3x - 3$

6. $y = -3x^2 + 2x + 1$
$y = 5 - 3x$

Solve the system by elimination.

7. $y = x^2 + x$
$y = x + 5$

8. $y = 9x^2 + 8x - 6$
$y = 5x - 4$

9. $y = 2x + 5$
$y = -3x^2 + x - 4$

Approximating Solutions

When you cannot find the exact solution(s) of a system of equations, you can analyze output values to approximate the solution(s).

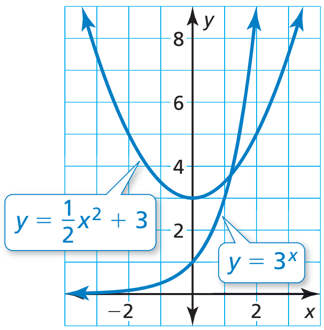
EXAMPLE 4 Approximating Solutions of a Nonlinear System

Approximate the solution(s) of the system to the nearest thousandth.

$y = \frac{1}{2}x^2 + 3$ Equation 1

$y = 3^x$ Equation 2

SOLUTION



Sketch a graph of the system. You can see that the system has one solution between $x = 1$ and $x = 2$.

Substitute 3^x for y in Equation 1 and rewrite the equation.

$3^x = \frac{1}{2}x^2 + 3$ Substitute 3^x for y in Equation 1.

$3^x - \frac{1}{2}x^2 - 3 = 0$ Rewrite the equation.

Because you do not know how to solve this equation algebraically, let $f(x) = 3^x - \frac{1}{2}x^2 - 3$. Then evaluate the function for x-values between 1 and 2.

$f(1.1) \approx -0.26$
$f(1.2) \approx 0.02$
Because $f(1.1) < 0$ and $f(1.2) > 0$, the zero is between 1.1 and 1.2.

REMEMBER
The function values that are closest to 0 correspond to x-values that best approximate the zeros of the function.

$f(1.2)$ is closer to 0 than $f(1.1)$, so decrease your guess and evaluate $f(1.19)$.

$f(1.19) \approx -0.012$ Because $f(1.19) < 0$ and $f(1.2) > 0$, the zero is between 1.19 and 1.2. So, increase guess.

$f(1.191) \approx -0.009$ Result is negative. Increase guess.

$f(1.192) \approx -0.006$ Result is negative. Increase guess.

$f(1.193) \approx -0.003$ Result is negative. Increase guess.

$f(1.194) \approx -0.0002$ Result is negative. Increase guess.

$f(1.195) \approx 0.003$ Result is positive.

Because $f(1.194)$ is closest to 0, $x \approx 1.194$.

Substitute $x = 1.194$ into one of the original equations and solve for y.

$$y = \frac{1}{2}x^2 + 3 = \frac{1}{2}(1.194)^2 + 3 \approx 3.713$$

So, the solution of the system is about (1.194, 3.713).

Recall that you can use systems of equations to solve equations with variables on both sides. To solve $f(x) = g(x)$, graph the system of equations $y = f(x)$ and $y = g(x)$. The x-value of each solution of the system is a solution of $f(x) = g(x)$.

REMEMBER

When entering the equations, be sure to use an appropriate viewing window that shows all the points of intersection. For this system, an appropriate viewing window is $-4 \le x \le 4$ and $-4 \le y \le 4$.

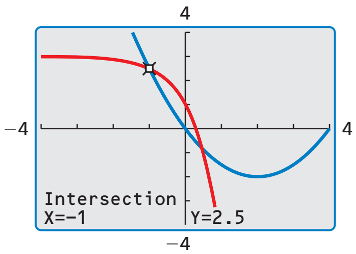
EXAMPLE 5 Approximating Solutions of an Equation

Solve $-2(4)^x + 3 = 0.5x^2 - 2x$.

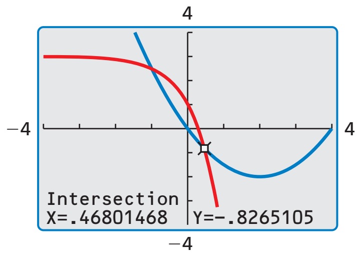
SOLUTION

You do not know how to solve this equation algebraically. So, use each side of the equation to write the system $y = -2(4)^x + 3$ and $y = 0.5x^2 - 2x$.

Method 1 Use a graphing calculator to graph the system. Then use the *intersect* feature to find the coordinates of each point of intersection.



One point of intersection is $(-1, 2.5)$.



The other point of intersection is about $(0.47, -0.83)$.

▶ So, the solutions of the equation are $x = -1$ and $x \approx 0.47$.

Method 2 Use the *table* feature to create a table of values for the equations. Find the x-values for which the corresponding y-values are approximately equal.

X	Y1	Y2
-1.03	2.5204	2.5905
-1.02	2.5137	2.5602
-1.01	2.5069	2.5301
-1	2.5	2.5
-.99	2.493	2.4701
-.98	2.4859	2.4402
-.97	2.4788	2.4105
X=-1		

When $x = -1$, the corresponding y-values are 2.5.

X	Y1	Y2
.44	-.6808	-.7832
.45	-.7321	-.7988
.46	-.7842	-.8142
.47	-.8371	-.8296
.48	-.8906	-.8448
.49	-.9449	-.86
.50	-1	-.875
X=.47		

When $x = 0.47$, the corresponding y-values are approximately -0.83.

STUDY TIP

You can use the differences between the corresponding y-values to determine the best approximation of a solution.

▶ So, the solutions of the equation are $x = -1$ and $x \approx 0.47$.

Monitoring Progress

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Use the method in Example 4 to approximate the solution(s) of the system to the nearest thousandth.

10. $y = 4^x$
$y = x^2 + x + 3$

11. $y = 4x^2 - 1$
$y = -2(3)^x + 4$

12. $y = x^2 + 3x$
$y = -x^2 + x + 10$

Solve the equation. Round your solution(s) to the nearest hundredth.

13. $3^x - 1 = x^2 - 2x + 5$

14. $4x^2 + x = -2\left(\frac{1}{2}\right)^x + 5$

4.8 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. VOCABULARY Describe how to use substitution to solve a system of nonlinear equations.

2. WRITING How is solving a system of nonlinear equations similar to solving a system of linear equations? How is it different?

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, match the system of equations with its graph. Then solve the system.

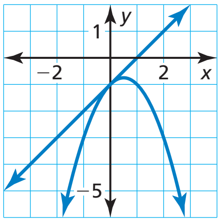
3. $y = x^2 - 2x + 1$
$y = x + 1$

4. $y = x^2 + 3x + 2$
$y = -x - 3$

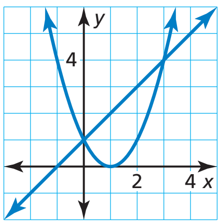
5. $y = x - 1$
$y = -x^2 + x - 1$

6. $y = -x + 3$
$y = -x^2 - 2x + 5$

A.



B.



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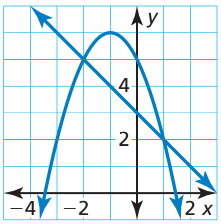

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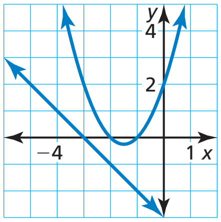


In Exercises 7–12, solve the system by graphing. *(See Example 1.)*

7. $y = 3x^2 - 2x + 1$
$y = x + 7$

8. $y = x^2 + 2x + 5$
$y = -2x - 5$

9. $y = -2x^2 - 4x$
$y = 2$

10. $y = \frac{1}{2}x^2 - 3x + 4$
$y = x - 2$

11. $y = \frac{1}{3}x^2 + 2x - 3$
$y = 2x$

12. $y = 4x^2 + 5x - 7$
$y = -3x + 5$

In Exercises 13–18, solve the system by substitution. *(See Example 2.)*

13. $y = x - 5$
$y = x^2 + 4x - 5$

14. $y = -3x^2$
$y = 6x + 3$

15. $y = -x + 7$
$y = -x^2 - 2x - 1$

16. $y = -x^2 + 7$
$y = 2x + 4$

17. $y - 5 = -x^2$
$y = 5$

18. $y = 2x^2 + 3x - 4$
$y - 4x = 2$

In Exercises 19–26, solve the system by elimination. *(See Example 3.)*

19. $y = x^2 - 5x - 7$
$y = -5x + 9$

20. $y = -3x^2 + x + 2$
$y = x + 4$

21. $y = -x^2 - 2x + 2$
$y = 4x + 2$

22. $y = -2x^2 + x - 3$
$y = 2x - 2$

23. $y = 2x - 1$
$y = x^2$

24. $y = x^2 + x + 1$
$y = -x - 2$

25. $y + 2x = 0$
$y = x^2 + 4x - 6$

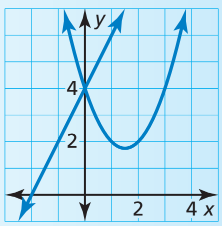
26. $y = 2x - 7$
$y + 5x = x^2 - 2$

27. ERROR ANALYSIS Describe and correct the error in solving the system of equations by graphing.

✗

$y = x^2 - 3x + 4$
$y = 2x + 4$

The only solution of the system is (0, 4).



28. **ERROR ANALYSIS** Describe and correct the error in solving for one of the variables in the system.

✗

$y = 3x^2 - 6x + 4$

$y = 4$

$y = 3(4)^2 - 6(4) + 4$ Substitute.

$y = 28$ Simplify.

In Exercises 29–32, use the table to describe the locations of the zeros of the quadratic function *f*.

29.

x	-4	-3	-2	-1	0	1
$f(x)$	-2	2	4	4	2	-2

30.

x	-1	0	1	2	3	4
$f(x)$	11	5	1	-1	-1	1

31.

x	-4	-3	-2	-1	0	1
$f(x)$	3	-1	-1	3	11	23

32.

x	1	2	3	4	5	6
$f(x)$	-25	-9	1	5	3	-5

In Exercises 33–38, use the method in Example 4 to approximate the solution(s) of the system to the nearest thousandth. *(See Example 4.)*

33. $y = x^2 + 2x + 3$
$y = 3^x$

34. $y = 2^x + 5$
$y = x^2 - 3x + 1$

35. $y = 2(4)^x - 1$
$y = 3x^2 + 8x$

36. $y = -x^2 - 4x - 4$
$y = -5^x - 2$

37. $y = -x^2 - x + 5$
$y = 2x^2 + 6x - 3$

38. $y = 2x^2 + x - 8$
$y = x^2 - 5$

In Exercises 39–46, solve the equation. Round your solution(s) to the nearest hundredth. *(See Example 5.)*

39. $3x + 1 = x^2 + 7x - 1$

40. $-x^2 + 2x = -2x + 5$

41. $x^2 - 6x + 4 = -x^2 - 2x$

42. $2x^2 + 8x + 10 = -x^2 - 2x + 5$

43. $-4\left(\frac{1}{2}\right)^x = -x^2 - 5$

44. $1.5(2)^x - 3 = -x^2 + 4x$

45. $8^{x-2} + 3 = 2\left(\frac{3}{2}\right)^x$

46. $-0.5(4)^x = 5^x - 6$

47. **COMPARING METHODS** Solve the system in Exercise 37 using substitution. Compare the exact solutions to the approximated solutions.

48. **COMPARING METHODS** Solve the system in Exercise 38 using elimination. Compare the exact solutions to the approximated solutions.

49. **MODELING WITH MATHEMATICS** The attendances y for two movies can be modeled by the following equations, where x is the number of days since the movies opened.

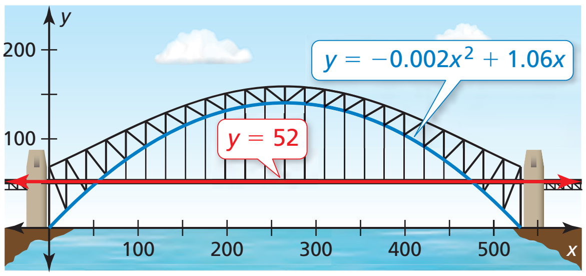
$y = -x^2 + 35x + 100$ Movie A

$y = -5x + 275$ Movie B

When is the attendance for each movie the same?

50. **MODELING WITH MATHEMATICS** You and a friend are driving boats on the same lake. Your path can be modeled by the equation $y = -x^2 - 4x - 1$, and your friend's path can be modeled by the equation $y = 2x + 8$. Do your paths cross each other? If so, what are the coordinates of the point(s) where the paths meet?

51. **MODELING WITH MATHEMATICS** The arch of a bridge can be modeled by $y = -0.002x^2 + 1.06x$, where x is the distance (in meters) from the left pylons and y is the height (in meters) of the arch above the water. The road can be modeled by the equation $y = 52$. To the nearest meter, how far from the left pylons are the two points where the road intersects the arch of the bridge?



52. **MAKING AN ARGUMENT** Your friend says that a system of equations consisting of a linear equation and a quadratic equation can have zero, one, two, or infinitely many solutions. Is your friend correct? Explain.

COMPARING METHODS **In Exercises 53 and 54, solve the system of equations by (a) graphing, (b) substitution, and (c) elimination. Which method do you prefer? Explain your reasoning.**

53. $y = 4x + 3$
$y = x^2 + 4x - 1$

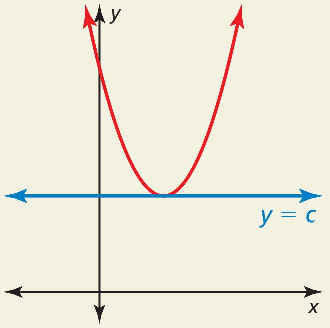
54. $y = x^2 - 5$
$y = -x + 7$

55. **MODELING WITH MATHEMATICS** The function $y = -x^2 + 65x + 256$ models the number y of subscribers to a website, where x is the number of days since the website launched. The number of subscribers to a competitor's website can be modeled by a linear function. The websites have the same number of subscribers on Days 1 and 34.

a. Write a linear function that models the number of subscribers to the competitor's website.

b. Solve the system to verify the function from part (a).

56. **HOW DO YOU SEE IT?** The diagram shows the graphs of two equations in a system that has one solution.



a. How many solutions will the system have when you change the linear equation to $y = c + 2$?

b. How many solutions will the system have when you change the linear equation to $y = c - 2$?

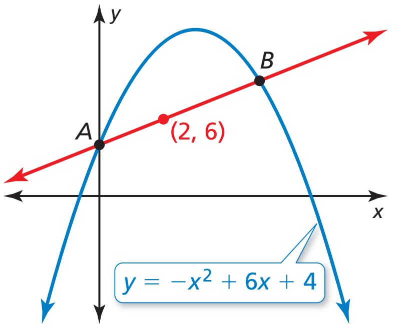
57. **WRITING** A system of equations consists of a quadratic equation whose graph opens up and a quadratic equation whose graph opens down. Describe the possible numbers of solutions of the system. Sketch examples to justify your answer.

58. **PROBLEM SOLVING** The population of a country is 2 million people and increases by 3% each year. The country's food supply is sufficient to feed 3 million people and increases at a constant rate that feeds 0.25 million additional people each year.

a. When will the country first experience a food shortage?

b. The country doubles the rate at which its food supply increases. Will food shortages still occur? If so, in what year?

59. **ANALYZING GRAPHS** Use the graphs of the linear and quadratic functions.



a. Find the coordinates of point A.

b. Find the coordinates of point B.

60. **THOUGHT PROVOKING** Is it possible for a system of two quadratic equations to have exactly three solutions? exactly four solutions? Explain your reasoning.

61. **PROBLEM SOLVING** Solve the system of three equations shown.

$y = 2x - 8$
$y = x^2 - 4x - 3$
$y = -3(2)^x$

62. **PROBLEM SOLVING** Find the point(s) of intersection, if any, of the line $y = -x - 1$ and the circle $x^2 + y^2 = 41$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Graph the inequality in a coordinate plane. *(Skills Review Handbook)*

63. $y \leq 4x - 1$ **64.** $y > -\frac{1}{2}x + 3$ **65.** $4y - 12 \geq 8x$ **66.** $3y + 3 < x$

Graph the function. Describe the domain and range. *(Section 3.3)*

67. $y = 3x^2 + 2$ **68.** $y = -x^2 - 6x$ **69.** $y = -2x^2 + 12x - 7$ **70.** $y = 5x^2 + 10x - 3$

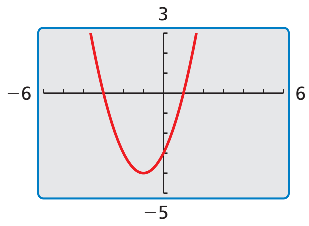
4.9 Quadratic Inequalities

Essential Question How can you solve a quadratic inequality?

EXPLORATION 1 **Solving a Quadratic Inequality**

Work with a partner. The graphing calculator screen shows the graph of

$$f(x) = x^2 + 2x - 3.$$



Explain how you can use the graph to solve the inequality

$$x^2 + 2x - 3 \leq 0.$$

Then solve the inequality.

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore your understanding of concepts.

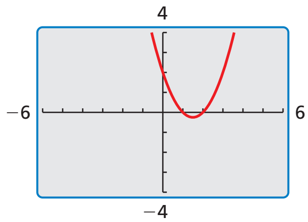
EXPLORATION 2 **Solving Quadratic Inequalities**

Work with a partner. Match each inequality with the graph of its related quadratic function. Then use the graph to solve the inequality.

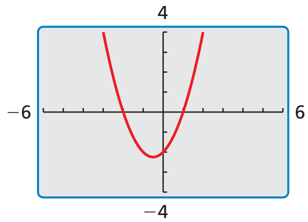
a. $x^2 - 3x + 2 > 0$ **b.** $x^2 - 4x + 3 \leq 0$ **c.** $x^2 - 2x - 3 < 0$

d. $x^2 + x - 2 \geq 0$ **e.** $x^2 - x - 2 < 0$ **f.** $x^2 - 4 > 0$

A.



B.



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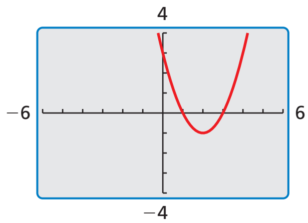

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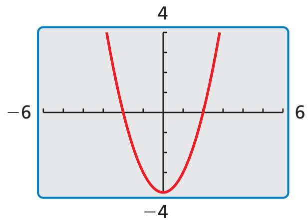
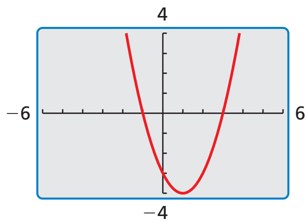
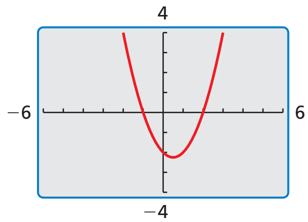


E.



F.



Communicate Your Answer

3. How can you solve a quadratic inequality?

4. Explain how you can use the graph in Exploration 1 to solve each inequality. Then solve each inequality.

a. $x^2 + 2x - 3 > 0$ **b.** $x^2 + 2x - 3 < 0$ **c.** $x^2 + 2x - 3 \geq 0$

4.9 Lesson

Core Vocabulary

quadratic inequality in two variables, *p. 260*
quadratic inequality in one variable, *p. 262*

Previous
linear inequality in two variables

What You Will Learn

- Graph quadratic inequalities in two variables.
- Solve quadratic inequalities in one variable.

Graphing Quadratic Inequalities in Two Variables

A **quadratic inequality in two variables** can be written in one of the following forms, where a, b, and c are real numbers and $a \neq 0$.

$$y < ax^2 + bx + c \qquad y > ax^2 + bx + c$$

$$y \leq ax^2 + bx + c \qquad y \geq ax^2 + bx + c$$

The graph of any such inequality consists of all solutions (x, y) of the inequality.

Previously, you graphed linear inequalities in two variables. You can use a similar procedure to graph quadratic inequalities in two variables.

Core Concept

Graphing a Quadratic Inequality in Two Variables

To graph a quadratic inequality in one of the forms above, follow these steps.

Step 1 Graph the parabola with the equation $y = ax^2 + bx + c$. Make the parabola *dashed* for inequalities with $<$ or $>$ and *solid* for inequalities with $\leq$ or $\geq$.

Step 2 Test a point (x, y) inside the parabola to determine whether the point is a solution of the inequality.

Step 3 Shade the region inside the parabola if the point from Step 2 is a solution. Shade the region outside the parabola if it is not a solution.

EXAMPLE 1 Graphing a Quadratic Inequality in Two Variables

Graph $y < -x^2 - 2x - 1$.

SOLUTION

Step 1 Graph $y = -x^2 - 2x - 1$. Because the inequality symbol is $<$, make the parabola dashed.

Step 2 Test a point inside the parabola, such as $(0, -3)$.

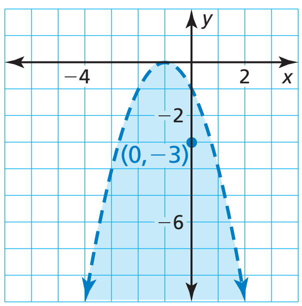
$$y < -x^2 - 2x - 1$$

$$-3 \stackrel{?}{<} -0^2 - 2(0) - 1$$

$$-3 < -1 \checkmark$$

So, $(0, -3)$ is a solution of the inequality.

Step 3 Shade the region inside the parabola.



LOOKING FOR STRUCTURE

Notice that testing a point is less complicated when the x-value is 0 (the point is on the y-axis).

EXAMPLE 2 Using a Quadratic Inequality in Real Life

A manila rope used for rappelling down a cliff can safely support a weight W (in pounds) provided

$$W \leq 1480d^2$$

where d is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.

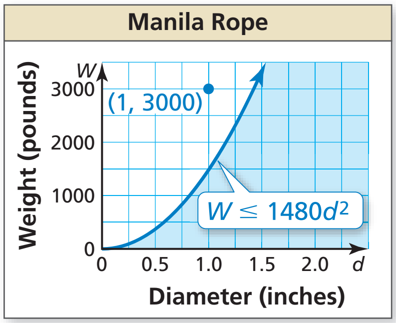
SOLUTION

Graph $W = 1480d^2$ for nonnegative values of d. Because the inequality symbol is $\leq$, make the parabola solid. Test a point inside the parabola, such as (1, 3000).

$$W \leq 1480d^2$$

$$3000 \stackrel{?}{\leq} 1480(1)^2$$

$$3000 \not\leq 1480$$



▶ Because (1, 3000) is not a solution, shade the region outside the parabola. The shaded region represents weights that can be supported by ropes with various diameters.

Graphing a *system* of quadratic inequalities is similar to graphing a system of linear inequalities. First graph each inequality in the system. Then identify the region in the coordinate plane common to all of the graphs. This region is called the *graph of the system*.

EXAMPLE 3 Graphing a System of Quadratic Inequalities

Graph the system of quadratic inequalities.

$y < -x^2 + 3$ Inequality 1

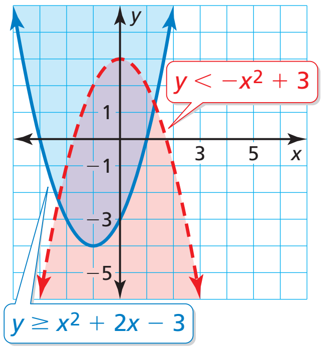
$y \geq x^2 + 2x - 3$ Inequality 2

SOLUTION

Step 1 Graph $y < -x^2 + 3$. The graph is the red region inside (but not including) the parabola $y = -x^2 + 3$.

Step 2 Graph $y \geq x^2 + 2x - 3$. The graph is the blue region inside and including the parabola $y = x^2 + 2x - 3$.

Step 3 Identify the purple region where the two graphs overlap. This region is the graph of the system.



Check

Check that a point in the solution region, such as (0, 0), is a solution of the system.

$$y < -x^2 + 3$$

$$0 \stackrel{?}{<} -0^2 + 3$$

$$0 < 3 \checkmark$$

$$y \geq x^2 + 2x - 3$$

$$0 \stackrel{?}{\geq} 0^2 + 2(0) - 3$$

$$0 \geq -3 \checkmark$$

Monitoring Progress

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Graph the inequality.

1. $y \geq x^2 + 2x - 8$ **2.** $y \leq 2x^2 - x - 1$ **3.** $y > -x^2 + 2x + 4$

4. Graph the system of inequalities consisting of $y \leq -x^2$ and $y > x^2 - 3$.

Solving Quadratic Inequalities in One Variable

A **quadratic inequality in one variable** can be written in one of the following forms, where a, b, and c are real numbers and $a \neq 0$.

$ax^2 + bx + c < 0 \qquad ax^2 + bx + c > 0 \qquad ax^2 + bx + c \leq 0 \qquad ax^2 + bx + c \geq 0$

You can solve quadratic inequalities using algebraic methods or graphs.

EXAMPLE 4 Solving a Quadratic Inequality Algebraically

Solve $x^2 - 3x - 4 < 0$ algebraically.

SOLUTION

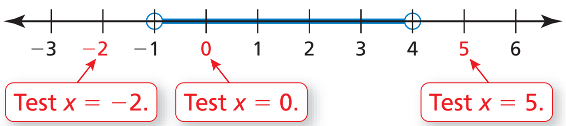
First, write and solve the equation obtained by replacing $<$ with $=$.

$x^2 - 3x - 4 = 0$ Write the related equation.

$(x - 4)(x + 1) = 0$ Factor.

$x = 4$ or $x = -1$ Zero-Product Property

The numbers -1 and 4 are the *critical values* of the original inequality. Plot -1 and 4 on a number line, using open dots because the values do not satisfy the inequality. The critical x-values partition the number line into three intervals. Test an x-value in each interval to determine whether it satisfies the inequality.



$(-2)^2 - 3(-2) - 4 = 6 \not< 0 \quad 0^2 - 3(0) - 4 = -4 < 0$ ✓ $\quad 5^2 - 3(5) - 4 = 6 \not< 0$

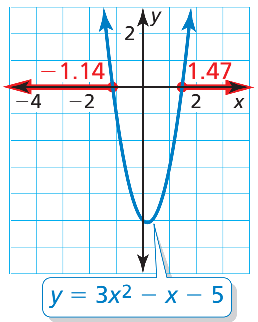
▶ So, the solution is $-1 < x < 4$.

Another way to solve $ax^2 + bx + c < 0$ is to first graph the related function $y = ax^2 + bx + c$. Then, because the inequality symbol is $<$, identify the x-values for which the graph lies *below* the x-axis. You can use a similar procedure to solve quadratic inequalities that involve $\leq$, $>$, or $\geq$.

EXAMPLE 5 Solving a Quadratic Inequality by Graphing

Solve $3x^2 - x - 5 \geq 0$ by graphing.

SOLUTION



The solution consists of the x-values for which the graph of $y = 3x^2 - x - 5$ lies on or above the x-axis. Find the x-intercepts of the graph by letting $y = 0$ and using the Quadratic Formula to solve $0 = 3x^2 - x - 5$ for x.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-5)}}{2(3)} \qquad a = 3, b = -1, c = -5$$

$$x = \frac{1 \pm \sqrt{61}}{6} \qquad \text{Simplify.}$$

The solutions are $x \approx -1.14$ and $x \approx 1.47$. Sketch a parabola that opens up and has -1.14 and 1.47 as x-intercepts. The graph lies on or above the x-axis to the left of (and including) $x = -1.14$ and to the right of (and including) $x = 1.47$.

▶ The solution of the inequality is approximately $x \leq -1.14$ or $x \geq 1.47$.

EXAMPLE 6 Modeling with Mathematics

A rectangular parking lot must have a perimeter of 440 feet and an area of at least 8000 square feet. Describe the possible lengths of the parking lot.

SOLUTION

1. **Understand the Problem** You are given the perimeter and the minimum area of a parking lot. You are asked to determine the possible lengths of the parking lot.
2. **Make a Plan** Use the perimeter and area formulas to write a quadratic inequality describing the possible lengths of the parking lot. Then solve the inequality.
3. **Solve the Problem** Let ℓ represent the length (in feet) and let w represent the width (in feet) of the parking lot.

$$\begin{array}{ll} \text{Perimeter} = 440 & \text{Area} \geq 8000 \\ 2\ell + 2w = 440 & \ell w \geq 8000 \end{array}$$

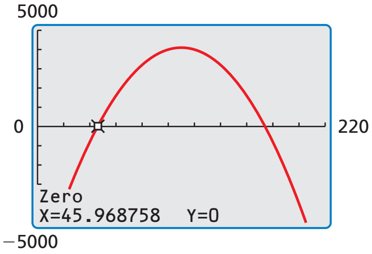
Solve the perimeter equation for w to obtain $w = 220 - \ell$. Substitute this into the area inequality to obtain a quadratic inequality in one variable.

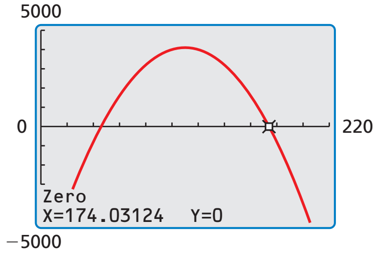
$$\begin{aligned} \ell w &\geq 8000 && \text{Write the area inequality.} \\ \ell(220 - \ell) &\geq 8000 && \text{Substitute } 220 - \ell \text{ for } w. \\ 220\ell - \ell^2 &\geq 8000 && \text{Distributive Property} \\ -\ell^2 + 220\ell - 8000 &\geq 0 && \text{Write in standard form.} \end{aligned}$$

ANOTHER WAY

You can graph each side of $220\ell - \ell^2 = 8000$ and use the intersection points to determine when $220\ell - \ell^2$ is greater than or equal to 8000.

Use a graphing calculator to find the ℓ-intercepts of $y = -\ell^2 + 220\ell - 8000$.





USING TECHNOLOGY

Variables displayed when using technology may not match the variables used in applications. In the graphs shown, the length ℓ corresponds to the independent variable x.

The ℓ-intercepts are $\ell \approx 45.97$ and $\ell \approx 174.03$. The solution consists of the ℓ-values for which the graph lies on or above the ℓ-axis. The graph lies on or above the ℓ-axis when $45.97 \leq \ell \leq 174.03$.

▶ So, the approximate length of the parking lot is at least 46 feet and at most 174 feet.

4. **Look Back** Choose a length in the solution region, such as $\ell = 100$, and find the width. Then check that the dimensions satisfy the original area inequality.

$$\begin{array}{rr} 2\ell + 2w = 440 & \ell w \geq 8000 \\ 2(100) + 2w = 440 & 100(120) \stackrel{?}{\geq} 8000 \\ w = 120 & 12{,}000 \geq 8000 \ \checkmark \end{array}$$

Monitoring Progress

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Solve the inequality.

5. $2x^2 + 3x \leq 2$ **6.** $-3x^2 - 4x + 1 < 0$ **7.** $2x^2 + 2 > -5x$

8. WHAT IF? In Example 6, the area must be at least 8500 square feet. Describe the possible lengths of the parking lot.

4.9 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

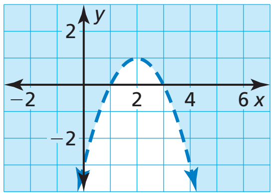
1. **WRITING** Compare the graph of a quadratic inequality in one variable to the graph of a quadratic inequality in two variables.

2. **WRITING** Explain how to solve $x^2 + 6x - 8 < 0$ using algebraic methods and using graphs.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, match the inequality with its graph. Explain your reasoning.

3. $y \leq x^2 + 4x + 3$
4. $y > -x^2 + 4x - 3$
5. $y < x^2 - 4x + 3$
6. $y \geq x^2 + 4x + 3$

A.


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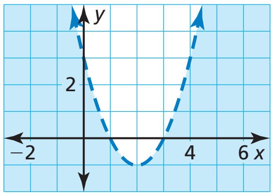

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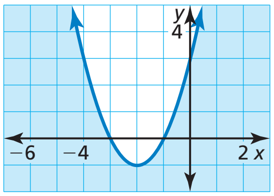

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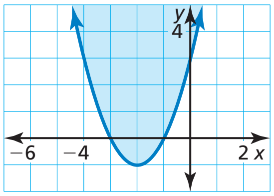
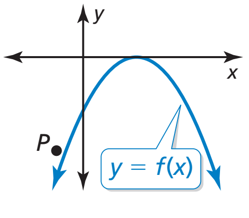
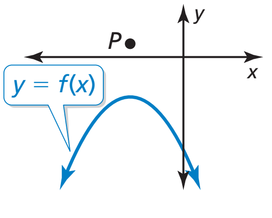


In Exercises 7–14, graph the inequality. *(See Example 1.)*

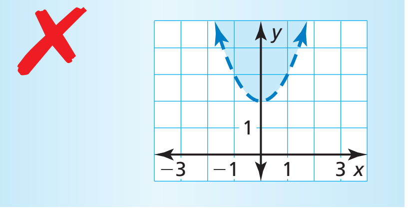
7. $y < -x^2$
8. $y \geq 4x^2$
9. $y > x^2 - 9$
10. $y < x^2 + 5$
11. $y \leq x^2 + 5x$
12. $y \geq -2x^2 + 9x - 4$
13. $y > 2(x + 3)^2 - 1$
14. $y \leq \left(x - \frac{1}{2}\right)^2 + \frac{5}{2}$

ANALYZING RELATIONSHIPS In Exercises 15 and 16, use the graph to write an inequality in terms of $f(x)$ so point P is a solution.

15.


16.


ERROR ANALYSIS In Exercises 17 and 18, describe and correct the error in graphing $y \geq x^2 + 2$.

17.


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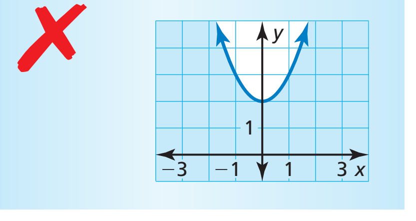


19. **MODELING WITH MATHEMATICS** A hardwood shelf in a wooden bookcase can safely support a weight W (in pounds) provided $W \leq 115x^2$, where x is the thickness (in inches) of the shelf. Graph the inequality and interpret the solution. *(See Example 2.)*

20. **MODELING WITH MATHEMATICS** A wire rope can safely support a weight W (in pounds) provided $W \leq 8000d^2$, where d is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.

In Exercises 21–26, graph the system of quadratic inequalities. *(See Example 3.)*

21. $y \geq 2x^2$
    $y < -x^2 + 1$

22. $y > -5x^2$
    $y > 3x^2 - 2$

23. $y \leq -x^2 + 4x - 4$
    $y < x^2 + 2x - 8$

24. $y \geq x^2 - 4$
    $y \leq -2x^2 + 7x + 4$

25. $y \geq 2x^2 + x - 5$
    $y < -x^2 + 5x + 10$

26. $y \geq x^2 - 3x - 6$
    $y \geq x^2 + 7x + 6$

In Exercises 27–34, solve the inequality algebraically. *(See Example 4.)*

27. $4x^2 < 25$

28. $x^2 + 10x + 9 < 0$

29. $x^2 - 11x \geq -28$

30. $3x^2 - 13x > -10$

31. $2x^2 - 5x - 3 \leq 0$

32. $4x^2 + 8x - 21 \geq 0$

33. $\frac{1}{2}x^2 - x > 4$

34. $-\frac{1}{2}x^2 + 4x \leq 1$

In Exercises 35–42, solve the inequality by graphing. *(See Example 5.)*

35. $x^2 - 3x + 1 < 0$

36. $x^2 - 4x + 2 > 0$

37. $x^2 + 8x > -7$

38. $x^2 + 6x < -3$

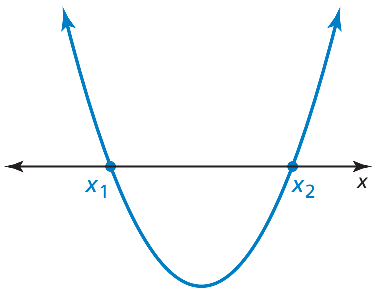
39. $3x^2 - 8 \leq -2x$

40. $3x^2 + 5x - 3 < 1$

41. $\frac{1}{3}x^2 + 2x \geq 2$

42. $\frac{3}{4}x^2 + 4x \geq 3$

43. DRAWING CONCLUSIONS Consider the graph of the function $f(x) = ax^2 + bx + c$.



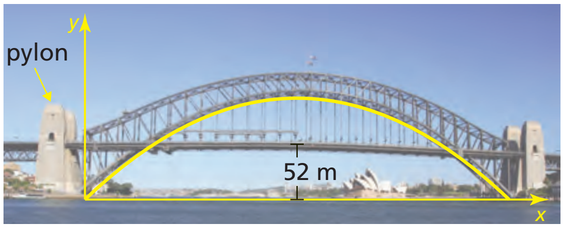
a. What are the solutions of $ax^2 + bx + c < 0$?

b. What are the solutions of $ax^2 + bx + c > 0$?

c. The graph of g represents a reflection in the x-axis of the graph of f. For which values of x is $g(x)$ positive?

44. MODELING WITH MATHEMATICS A rectangular fountain display has a perimeter of 400 feet and an area of at least 9100 feet. Describe the possible widths of the fountain. *(See Example 6.)*

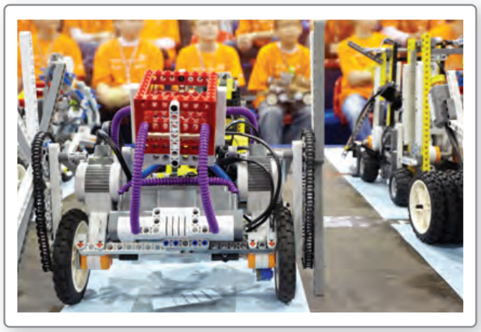
45. MODELING WITH MATHEMATICS The arch of the Sydney Harbor Bridge in Sydney, Australia, can be modeled by $y = -0.00211x^2 + 1.06x$, where x is the distance (in meters) from the left pylons and y is the height (in meters) of the arch above the water. For what distances x is the arch above the road?



46. PROBLEM SOLVING The number T of teams that have participated in a robot-building competition for high-school students over a recent period of time x (in years) can be modeled by

$$T(x) = 17.155x^2 + 193.68x + 235.81, \ 0 \leq x \leq 6.$$

After how many years is the number of teams greater than 1000? Justify your answer.



47. PROBLEM SOLVING A study found that a driver's reaction time $A(x)$ to audio stimuli and his or her reaction time $V(x)$ to visual stimuli (both in milliseconds) can be modeled by

$$A(x) = 0.0051x^2 - 0.319x + 15, \ 16 \leq x \leq 70$$

$$V(x) = 0.005x^2 - 0.23x + 22, \ 16 \leq x \leq 70$$

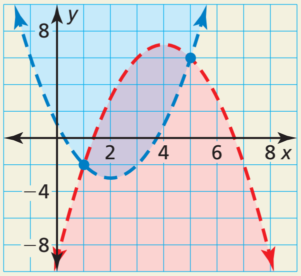
where x is the age (in years) of the driver.

a. Write an inequality that you can use to find the x-values for which $A(x)$ is less than $V(x)$.

b. Use a graphing calculator to solve the inequality $A(x) < V(x)$. Describe how you used the domain $16 \leq x \leq 70$ to determine a reasonable solution.

c. Based on your results from parts (a) and (b), do you think a driver would react more quickly to a traffic light changing from green to yellow or to the siren of an approaching ambulance? Explain.

48. **HOW DO YOU SEE IT?** The graph shows a system of quadratic inequalities.



a. Identify two solutions of the system.

b. Are the points $(1, -2)$ and $(5, 6)$ solutions of the system? Explain.

c. Is it possible to change the inequality symbol(s) so that one, but not both of the points in part (b), is a solution of the system? Explain.

49. **MODELING WITH MATHEMATICS** The length L (in millimeters) of the larvae of the black porgy fish can be modeled by

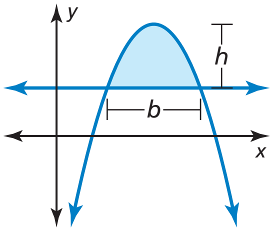
$$L(x) = 0.00170x^2 + 0.145x + 2.35,\ 0 \le x \le 40$$

where x is the age (in days) of the larvae. Write and solve an inequality to find at what ages a larva's length tends to be greater than 10 millimeters. Explain how the given domain affects the solution.

50. **MAKING AN ARGUMENT** You claim the system of inequalities below, where a and b are real numbers, has no solution. Your friend claims the system will always have at least one solution. Who is correct? Explain.

$$y < (x + a)^2$$
$$y < (x + b)^2$$

51. **MATHEMATICAL CONNECTIONS** The area A of the region bounded by a parabola and a horizontal line can be modeled by $A = \frac{2}{3}bh$, where b and h are as defined in the diagram. Find the area of the region determined by each pair of inequalities.



a. $y \le -x^2 + 4x$
$y \ge 0$

b. $y \ge x^2 - 4x - 5$
$y \le 7$

52. **THOUGHT PROVOKING** Draw a company logo that is created by the intersection of two quadratic inequalities. Justify your answer.

53. **REASONING** A truck that is 11 feet tall and 7 feet wide is traveling under an arch. The arch can be modeled by $y = -0.0625x^2 + 1.25x + 5.75$, where x and y are measured in feet.

a. Will the truck fit under the arch? Explain.

b. What is the maximum width that a truck 11 feet tall can have and still make it under the arch?

c. What is the maximum height that a truck 7 feet wide can have and still make it under the arch?

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Tell whether the function has a minimum value or maximum value. Then find the value. *(Section 3.3)*

54. $f(x) = -x^2 - 6x - 10$

55. $f(x) = -x^2 - 4x + 21$

56. $h(x) = \frac{1}{2}x^2 + 2x + 2$

57. $h(x) = x^2 + 3x - 18$

Find the zeros of the function. *(Section 3.5)*

58. $f(x) = (x + 7)(x - 9)$

59. $g(x) = x^2 - 4x$

60. $h(x) = -x^2 + 5x - 6$

4.6–4.9 What Did You Learn?

Core Vocabulary

imaginary unit *i*, *p. 238*
complex number, *p. 238*
imaginary number, *p. 238*
pure imaginary number, *p. 238*
complex conjugates, *p. 241*
system of nonlinear equations, *p. 252*
quadratic inequality in two variables, *p. 260*
quadratic inequality in one variable, *p. 262*

Core Concepts

Section 4.6

The Square Root of a Negative Number, *p. 238*
Sums and Differences of Complex Numbers, *p. 239*
Multiplying Complex Numbers, *p. 240*

Section 4.7

Finding Solutions and Zeros, *p. 246*
Using the Discriminant, *p. 247*

Section 4.8

Solving Nonlinear Systems, *p. 252*
Approximating Solutions, *p. 254*

Section 4.9

Graphing a Quadratic Inequality in Two Variables, *p. 260*
Solving Quadratic Inequalities in One Variable, *p. 262*

Mathematical Practices

1. How can you use technology to determine whose rocket lands first in part (b) of Exercise 36 on page 250?

2. Compare the methods used to solve Exercise 53 on page 258. Discuss the similarities and differences among the methods.

3. Explain your plan to find the possible widths of the fountain in Exercise 44 on page 265.

Performance Task:

The Golden Ratio

The golden ratio is one of the most famous numbers in history. It can be seen in the simplicity of a seashell as well as in the grandeur of the Greek Parthenon. What is this number and how can it help you decorate your room?

To explore the answers to these questions and more, check out the Performance Task and Real-Life STEM video at ***BigIdeasMath.com***.

4 Chapter Review

Dynamic Solutions available at *BigIdeasMath.com*

4.1 Properties of Radicals *(pp. 191–200)*

a. Simplify $\sqrt[3]{27x^{10}}$.

$$\sqrt[3]{27x^{10}} = \sqrt[3]{27 \cdot x^9 \cdot x}$$ Factor using the greatest perfect cube factors.

$$= \sqrt[3]{27} \cdot \sqrt[3]{x^9} \cdot \sqrt[3]{x}$$ Product Property of Cube Roots

$$= 3x^3\sqrt[3]{x}$$ Simplify.

b. Simplify $\frac{12}{3+\sqrt{5}}$.

$$\frac{12}{3+\sqrt{5}} = \frac{12}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}}$$ The conjugate of $3 + \sqrt{5}$ is $3 - \sqrt{5}$.

$$= \frac{12\left(3-\sqrt{5}\right)}{3^2 - \left(\sqrt{5}\right)^2}$$ Sum and difference pattern

$$= \frac{36 - 12\sqrt{5}}{4}$$ Simplify.

$$= 9 - 3\sqrt{5}$$ Simplify.

Simplify the expression.

1. $\sqrt{72p^7}$ **2.** $\sqrt{\frac{45}{7y}}$ **3.** $\sqrt[3]{\frac{125x^{11}}{4}}$ **4.** $\frac{8}{\sqrt{6}+2}$

5. $4\sqrt{3} + 5\sqrt{12}$ **6.** $15\sqrt[3]{2} - 2\sqrt[3]{54}$ **7.** $\left(3\sqrt{7} + 5\right)^2$ **8.** $\sqrt{6}\left(\sqrt{18} + \sqrt{8}\right)$

4.2 Solving Quadratic Equations by Graphing *(pp. 201–208)*

Solve $x^2 + 3x = 4$ by graphing.

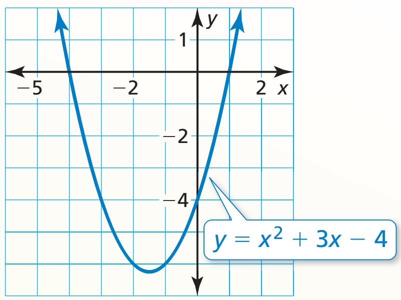
Step 1 Write the equation in standard form.

$x^2 + 3x = 4$ Write original equation.

$x^2 + 3x - 4 = 0$ Subtract 4 from each side.

Step 2 Graph the related function $y = x^2 + 3x - 4$.

Step 3 Find the x-intercepts. The x-intercepts are -4 and 1.



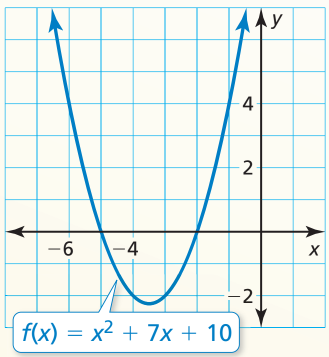
▶ So, the solutions are $x = -4$ and $x = 1$.

Solve the equation by graphing.

9. $x^2 - 9x + 18 = 0$ **10.** $x^2 - 2x = -4$

11. $-8x - 16 = x^2$

12. The graph of $f(x) = x^2 + 7x + 10$ is shown. Find the zeros of f.

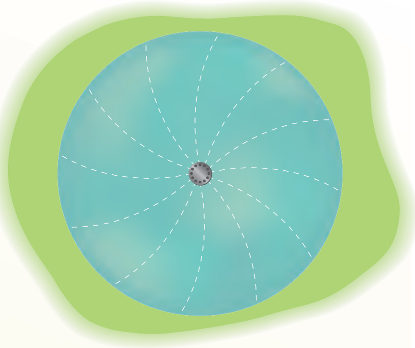


13. Graph $f(x) = x^2 + 2x - 5$. Approximate the zeros of f to the nearest tenth.

4.3 Solving Quadratic Equations Using Square Roots (pp. 209–214)

A sprinkler sprays water that covers a circular region of 90π square feet. Find the diameter of the circle.

Write an equation using the formula for the area of a circle.



$A = \pi r^2$	Write the formula.
$90\pi = \pi r^2$	Substitute 90π for A.
$90 = r^2$	Divide each side by π.
$\pm\sqrt{90} = r$	Take the square root of each side.
$\pm 3\sqrt{10} = r$	Simplify.

A diameter cannot be negative, so use the positive square root. The diameter is twice the radius. So, the diameter is $6\sqrt{10}$.

▶ The diameter of the circle is $6\sqrt{10} \approx 19$ feet.

Solve the equation using square roots. Round your solutions to the nearest hundredth, if necessary.

14. $x^2 + 5 = 17$

15. $x^2 - 14 = -14$

16. $(x + 2)^2 = 64$

17. $4x^2 + 25 = 75$

18. $(x - 1)^2 = 0$

19. $19 = 30 - 5x^2$

4.4 Solving Quadratic Equations by Completing the Square (pp. 215–224)

Solve $x^2 - 6x + 4 = 11$ by completing the square.

$x^2 - 6x + 4 = 11$	Write the equation.
$x^2 - 6x = 7$	Subtract 4 from each side.
$x^2 - 6x + (-3)^2 = 7 + (-3)^2$	Complete the square by adding $\left(\frac{-6}{2}\right)^2$, or $(-3)^2$, to each side.
$(x - 3)^2 = 16$	Write the left side as the square of a binomial.
$x - 3 = \pm 4$	Take the square root of each side.
$x = 3 \pm 4$	Add 3 to each side.

▶ The solutions are $x = 3 + 4 = 7$ and $x = 3 - 4 = -1$.

Solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary.

20. $x^2 + 6x - 40 = 0$

21. $x^2 + 2x + 5 = 4$

22. $2x^2 - 4x = 10$

Determine whether the quadratic function has a maximum or minimum value. Then find the value.

23. $y = -x^2 + 6x - 1$

24. $f(x) = x^2 + 4x + 11$

25. $y = 3x^2 - 24x + 15$

26. The width w of a credit card is 3 centimeters shorter than the length ℓ. The area is 46.75 square centimeters. Find the perimeter.

4.5 Solving Quadratic Equations Using the Quadratic Formula (pp. 225–234)

Solve $-3x^2 + x = -8$ using the Quadratic Formula.

$$-3x^2 + x = -8$$ Write the equation.

$$-3x^2 + x + 8 = 0$$ Write in standard form.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$x = \frac{-1 \pm \sqrt{1^2 - 4(-3)(8)}}{2(-3)}$$ Substitute -3 for a, 1 for b, and 8 for c.

$$x = \frac{-1 \pm \sqrt{97}}{-6}$$ Simplify.

▶ So, the solutions are $x = \frac{-1 + \sqrt{97}}{-6} \approx -1.5$ and $x = \frac{-1 - \sqrt{97}}{-6} \approx 1.8$.

Solve the equation using the Quadratic Formula. Round your solutions to the nearest tenth, if necessary.

27. $x^2 + 2x - 15 = 0$ **28.** $2x^2 - x + 8 = 16$ **29.** $-5x^2 + 10x = 5$

Find the number of *x*-intercepts of the graph of the function.

30. $y = -x^2 + 6x - 9$ **31.** $y = 2x^2 + 4x + 8$ **32.** $y = -\frac{1}{2}x^2 + 2x$

4.6 Complex Numbers (pp. 237–244)

Perform each operation. Write the answer in standard form.

a. $(3 - 6i) - (7 + 2i) = (3 - 7) + (-6 - 2)i$ Definition of complex subtraction

$= -4 - 8i$ Write in standard form.

b. $5i(4 + 5i) = 20i + 25i^2$ Distributive Property

$= 20i + 25(-1)$ Use $i^2 = -1$.

$= -25 + 20i$ Write in standard form.

c. $(2 + 5i)(1 - 3i) = 2 - 6i + 5i - 15i^2$ Multiply using FOIL.

$= 2 - i - 15(-1)$ Simplify and use $i^2 = -1$.

$= 2 - i + 15$ Simplify.

$= 17 - i$ Write in standard form.

33. Find the values of x and y that satisfy the equation $36 - yi = 4x + 3i$.

Perform the operation. Write the answer in standard form.

34. $(-2 + 3i) + (7 - 6i)$ **35.** $(9 + 3i) - (-2 - 7i)$ **36.** $(5 + 6i)(-4 + 7i)$

37. Multiply $8 - 2i$ by its complex conjugate.

4.7 Solving Quadratic Equations with Complex Solutions (pp. 245–250)

Solve each equation.

a. $x^2 - 14x + 53 = 0$

The coefficient of the x^2-term is 1, and the coefficient of the x-term is an even number. So, solve by completing the square.

$x^2 - 14x + 53 = 0$	Write the equation.
$x^2 - 14x = -53$	Subtract 53 from each side.
$x^2 - 14x + 49 = -53 + 49$	Complete the square for $x^2 - 14x$.
$(x - 7)^2 = -4$	Write the left side as the square of a binomial.
$x - 7 = \pm\sqrt{-4}$	Take the square root of each side.
$x = 7 \pm \sqrt{-4}$	Add 7 to each side.
$x = 7 \pm 2i$	Simplify the radical.

▶ The solutions are $7 + 2i$ and $7 - 2i$.

b. $9x^2 - 6x + 5 = 0$

The equation is not factorable, and completing the square would result in fractions. So, solve using the Quadratic Formula.

$x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(5)}}{2(9)}$	Substitute 9 for *a*, −6 for *b*, and 5 for *c*.
$x = \dfrac{6 \pm \sqrt{-144}}{18}$	Simplify.
$x = \dfrac{6 \pm 12i}{18}$	Write in terms of *i*.
$x = \dfrac{1 \pm 2i}{3}$	Simplify.

▶ The solutions are $\dfrac{1 + 2i}{3}$ and $\dfrac{1 - 2i}{3}$.

Solve the equation using any method. Explain your choice of method.

38. $-x^2 - 100 = 0$

39. $4x^2 + 53 = -11$

40. $x^2 + 16x - 17 = 0$

41. $-2x^2 + 12x = 36$

42. $2x^2 + 5x = 4$

43. $3x^2 - 7x + 13 = 0$

Find the zeros of the function.

44. $f(x) = x^2 + 81$

45. $f(x) = x^2 - 2x + 9$

46. $f(x) = -8x^2 + 4x - 1$

47. While marching, a drum major tosses a baton into the air from an initial height of 6 feet. The baton has an initial vertical velocity of 32 feet per second. Does the baton reach a height of 25 feet? 20 feet? Explain your reasoning.

4.8 Solving Nonlinear Systems of Equations (pp. 251–258)

Solve the system by substitution. $\quad y = x^2 - 5 \quad$ Equation 1
$\quad y = -x + 1 \quad$ Equation 2

Step 1 The equations are already solved for y.

Step 2 Substitute $-x + 1$ for y in Equation 1 and solve for x.

$$-x + 1 = x^2 - 5 \qquad \text{Substitute } -x + 1 \text{ for } y \text{ in Equation 1.}$$
$$1 = x^2 + x - 5 \qquad \text{Add } x \text{ to each side.}$$
$$0 = x^2 + x - 6 \qquad \text{Subtract 1 from each side.}$$
$$0 = (x + 3)(x - 2) \qquad \text{Factor the polynomial.}$$
$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0 \qquad \text{Zero-Product Property}$$
$$x = -3 \quad \text{or} \quad x = 2 \qquad \text{Solve for } x.$$

Step 3 Substitute -3 and 2 for x in Equation 2 and solve for y.

$$y = -(-3) + 1 \qquad \text{Substitute for } x \text{ in Equation 2.} \qquad y = -2 + 1$$
$$= 4 \qquad \text{Simplify.} \qquad = -1$$

▶ So, the solutions are $(-3, 4)$ and $(2, -1)$.

Solve the system using any method.

48. $y = x^2 - 2x - 4$
$y = -5$

49. $y = x^2 - 9$
$y = 2x + 5$

50. $y = 2\left(\frac{1}{2}\right)^x - 5$
$y = -x^2 - x + 4$

4.9 Quadratic Inequalities (pp. 259–266)

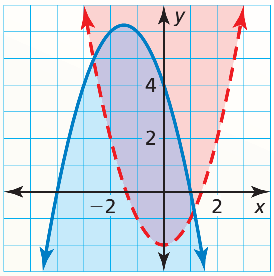
Graph the system of quadratic inequalities.

$y > x^2 - 2 \quad$ Inequality 1
$y \leq -x^2 - 3x + 4 \quad$ Inequality 2

Step 1 Graph $y > x^2 - 2$. The graph is the red region inside (but not including) the parabola $y = x^2 - 2$.

Step 2 Graph $y \leq -x^2 - 3x + 4$. The graph is the blue region inside and including the parabola $y = -x^2 - 3x + 4$.

Step 3 Identify the purple region where the two graphs overlap. This region is the graph of the system.



Graph the inequality.

51. $y > x^2 + 8x + 16$

52. $y \geq x^2 + 6x + 8$

53. $x^2 + y \leq 7x - 12$

Graph the system of quadratic inequalities.

54. $x^2 - 4x + 8 > y$
$-x^2 + 4x + 2 \leq y$

55. $2x^2 - x \geq y - 5$
$0.5x^2 > y - 2x - 1$

56. $-3x^2 - 2x \leq y + 1$
$-2x^2 + x - 5 > -y$

57. Solve (a) $3x^2 + 2 \leq 5x$ and (b) $-x^2 - 10x < 21$.

4 Chapter Test

Solve the equation using any method. Explain your choice of method.

1. $x^2 + 121 = 0$

2. $x^2 - 6x = -58$

3. $-2x^2 + 3x + 7 = 0$

4. $x^2 - 7x + 12 = 0$

5. $5x^2 + x + 4 = 0$

6. $(4x + 3)^2 = 16$

Perform the operation. Write the answer in standard form.

7. $(2 + 5i) + (-4 + 3i)$

8. $(3 + 9i) - (1 - 7i)$

9. $(2 + 4i)(-3 - 5i)$

Solve the system using any method.

10. $y = x^2 - 4x - 2$
$y = -4x + 2$

11. $y = -5x^2 + x - 1$
$y = -7$

12. $y = \frac{1}{2}(4)^x + 1$
$y = x^2 - 2x + 4$

13. Graph the system of inequalities consisting of $y \geq 2x^2 - 3$ and $y < -x^2 + x + 4$.

14. Write an expression involving radicals in which a conjugate can be used to simplify the expression. Then simplify the expression.

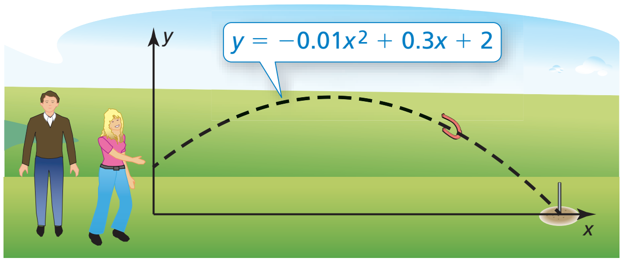
15. Describe the value(s) of c for which $x^2 + 4x = c$ has (a) two real solutions, (b) one real solution, and (c) two imaginary solutions.

16. Simplify the expression $\sqrt{30x^7} \cdot \dfrac{36}{\sqrt{3}}$.

17. A skier leaves an 8-foot-tall ramp with an initial vertical velocity of 28 feet per second. The skier has a perfect landing. How many points does the skier earn?

Criteria	Scoring
Maximum height	1 point per foot
Time in air	5 points per second
Perfect landing	25 points

18. You are playing a game of horseshoes. One of your tosses is modeled in the diagram, where x is the horseshoe's horizontal position (in feet) and y is the corresponding height (in feet). Does the horseshoe reach a height of 8 feet? 4 feet? Explain your reasoning.



19. The numbers y of two types of bacteria after x hours are represented by the models below.

$y = 3x^2 + 8x + 20$ Type A
$y = 27x + 60$ Type B

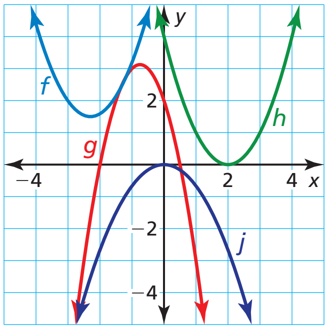
a. When are there 400 Type A bacteria?

b. When are the number of Type A and Type B bacteria the same?

c. When are there more Type A bacteria than Type B? When are there more Type B bacteria than Type A? Use a graph to support your answer.

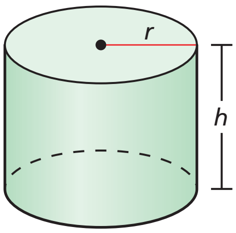
4 Cumulative Assessment

1. The graphs of four quadratic functions are shown. Determine whether the discriminants of the equations $f(x) = 0$, $g(x) = 0$, $h(x) = 0$, and $j(x) = 0$ are positive, negative, or zero.

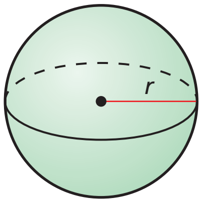


2. Your friend claims to be able to find the radius r of each figure, given the surface area S. Do you support your friend's claim? Justify your answer.

a. $S = 2\pi r^2 + 2\pi rh$



b. $S = 4\pi r^2$



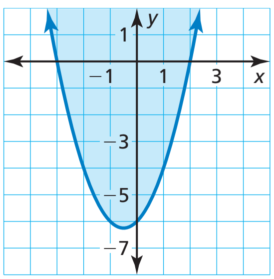
3. Choose values for the constants h and k in the equation $x = \frac{1}{4}(y - k)^2 + h$ so that each statement is true.

a. The graph of $x = \frac{1}{4}\left(y - \square\right)^2 + \square$ is a parabola with its vertex in the second quadrant.

b. The graph of $x = \frac{1}{4}\left(y - \square\right)^2 + \square$ is a parabola with its focus in the first quadrant.

c. The graph of $x = \frac{1}{4}\left(y - \square\right)^2 + \square$ is a parabola with its focus in the third quadrant.

4. The graph of which inequality is shown?



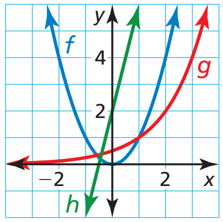
Ⓐ $y > x^2 + x - 6$

Ⓑ $y \geq x^2 + x - 6$

Ⓒ $y > x^2 - x - 6$

Ⓓ $y \geq x^2 - x - 6$

5. Use the graphs of the functions to answer each question.



a. Are there any values of x greater than 0 where $f(x) > h(x)$? Explain.

b. Are there any values of x greater than 1 where $g(x) > f(x)$? Explain.

c. Are there any values of x greater than 0 where $g(x) > h(x)$? Explain.

6. Which statement best describes the solution(s) of the system of equations?

$$y = x^2 + 2x - 8$$
$$y = 5x + 2$$

Ⓐ The graphs intersect at one point, $(-2, -8)$. So, there is one solution.

Ⓑ The graphs intersect at two points, $(-2, -8)$ and $(5, 27)$. So, there are two solutions.

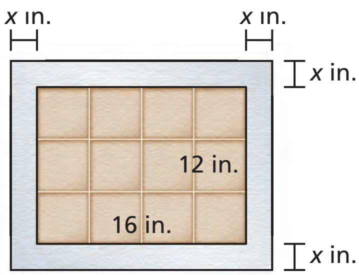
Ⓒ The graphs do not intersect. So, there is no solution.

Ⓓ The graph of $y = x^2 + 2x - 8$ has two x-intercepts. So, there are two solutions.

7. Which expressions are in simplest form?

$x\sqrt{45x}$	$\dfrac{16}{\sqrt{5}}$	$\sqrt[3]{\dfrac{4}{9}}$	$16\sqrt{5}$	$3x\sqrt{5x}$
$\dfrac{\sqrt[3]{x^4}}{2}$	$\dfrac{4\sqrt{7}}{3}$	$\dfrac{\sqrt{16}}{5}$	$2\sqrt[3]{x^2}$	$3\dfrac{\sqrt{7}}{\sqrt{x}}$

8. You are making a tabletop with a tiled center and a uniform mosaic border.



a. Write the polynomial in standard form that represents the perimeter of the tabletop.

b. Write the polynomial in standard form that represents the area of the tabletop.

c. The perimeter of the tabletop is less than 80 inches, and the area of tabletop is at least 252 square inches. Select all the possible values of x.

0.5	1	1.5	2	2.5	3	3.5	4

9. Let $f(x) = 2x - 5$. Solve $y = f(x)$ for x. Then find the input when the output is -4.